Linear Algebra (MATH 3333 - 04) Spring 2011
Homework 3

Due: Fri. Feb. 18, start of class

Instructions: Please read the homework policies and guidelines posted on the course webpage. You may **not** use a calculator (or computer). Make sure to write your name, course and section numbers in the top right corner of your solution set, as well as the assignment number on top. Please staple your homework. Sections and exercises refer to the exercises in the required course text.

Conceptual Questions (not to be turned in)

1. Why do we want to study linear transformations?
2. What is the benefit of using matrices to represent linear transformations?
3. Why do we want to be able to write down a matrix for rotation by θ or reflection about $y = mx$?

Written Assignment

Total: 100 points

Problem A. (20 points) Describe as best you can what the following linear transformations do geometrically:

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T_3 = T_1 \circ T_2, \quad T_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_5 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Problem B. (20 points) Write a matrix for the following transformations of $\mathbb{R}^2$:
(i) counter-clockwise rotation by 30 degrees,
(ii) clockwise rotation by 45 degrees,
(iii) reflection about $y = -2x$,
(iv) scaling by 3 in the x-direction and scaling by 4 in the y-direction.

Problem C. (10 points) Let S be the transformation from B(ii) and T be the transformation from B(iii). Write in matrix notation the transformations $S \circ T$ and $T \circ S$. Describe what they represent geometrically.

Problem D. (10 points) Recall from lecture that the determinant of a 2×2 matrix (or the corresponding linear transformation) is given by

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

(i) Also recall that reflection of $\mathbb{R}^2$ about the line $y = mx$ is given by $\frac{1}{m^2+1} \begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$. Check that the determinant of reflection about $y = mx$ is -1 as asserted in class.

(ii) There is one other reflection about a line through the origin: the one about the y-axis. Write down a matrix for it, and check it also has determinant -1.

Problem E. (10 points) We stated in class that any isometry, i.e. a linear transformation which preserves distances, has determinant ± 1. (Rotations are $+1$ and reflections, by the previous problem, are -1.) The converse (other) direction is not true: Find a 2×2 matrix whose determinant is ± 1 but is not an isometry.

Problem F. (10 points) Find all linear transformations (in matrix form) $T : \mathbb{R}^2 \to \mathbb{R}^2$ which preserve the unit square with vertices $(0,0), (1,0), (0,1)$ and $(1,1)$. (This means the image of the square under T is itself—it does not mean that $T(p) = p$ for any point p on the unit square.) Justify that your list is complete.

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Problem G. (20 points)

(i) Let $T_1(x, y) = (ex + fy, gx + hy)$ and $T_2(x, y) = (ax + by, cx + dy)$. Compute the composition $(T_1 \circ T_2)(x, y) = T_1(T_2(x, y))$ *without* using matrices.

(ii) Now writing T_1 and T_2 as matrices, compute the matrix multiplication $T_1 \cdot T_2$ and show that this product is the composition $T_1 \circ T_2$ from part (i).