(0) Chain Rule!! (a) Let \( z = f(u) \), where \( u = u(x, y) \). What is \( \frac{\partial z}{\partial x} \)?

(b) Let \( z = f(x, u) \), where \( u = u(x, y) \). What is \( \frac{\partial z}{\partial x} \)?

(1) Finding limits. For example, #12, #16 of §14.2.

(2) (Implicit differentiation) The relation \( x^2 + y^2 - z^2 = 2x(y + z) \) defines \( z \) as a function of \( x \) and \( y \) implicitly. Find \( \frac{\partial z}{\partial y} \).

(3) Find the equation of the tangent plane and the normal line to the surface \( z = \sin(x + y) \) at \((1, -1, 0)\).

(4) Directional derivative. You must first change the direction vector to a unit vector. Remember the formula \( D_u f(a, b) = u \cdot \nabla f(a, b) \).

(5) (Approximation by differential) Use differential to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.

(6) Suppose \( S \) is a surface with equation \( F(x, y, z) = k \), that is, it is a level surface of a function \( F \) of three variables, and let \( P(x_0, y_0, z_0) \) be a point on \( S \). Let \( C \) be any curve that lies on the surface \( S \) and passes through the point \( P \). Recall that the curve \( C \) is described by a continuous vector function \( r(t) = (x(t), y(t), z(t)) \). Let \( t_0 \) be the parameter value corresponding to \( P \); that is, \( r(t_0) = (x_0, y_0, z_0) \). Prove that the gradient \( \nabla F(x_0, y_0, z_0) \) is perpendicular to the tangent vector \( r'(t_0) \) at the point \( P \).

(7) If \( z = f(x, y) \), where \( x = s + t \) and \( y = s - t \), show that
\[
\left( \frac{\partial z}{\partial x} \right)^2 - \left( \frac{\partial z}{\partial y} \right)^2 = \frac{\partial z}{\partial s} \frac{\partial z}{\partial t}
\]

(8) Find all critical points of the function \( f(x, y) = 4xy - x^4 - y^4 - 1 \), and classify them.

(9) Setting up simultaneous equations from Lagrange multiplier method with two constraints.

(10) Find the extreme values of \( f(x, y) = e^{-xy} \) on the region \( x^2 + 4y^2 \leq 1 \).
(To analyze the boundary, use the Lagrange multiplier method).
Hints:

0. (a) \( \frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} = f'(u) \cdot \frac{\partial u}{\partial x} \) (b) \( \frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} \)

1. #12: Limit does not exist, #16: Limit is 0.

2. \( \frac{\partial z}{\partial y} = \frac{y-x}{x+z} \)

3. Change the equation to \( f(x, y, z) = \sin(x + y) - z = 0 \). Then \( \nabla f(1, -1, 0) = \langle 1, 1, -1 \rangle \). Thus, \( (x - 1) + (y + 1) - (z - 0) = 0 \) and \( \frac{x-1}{1} = \frac{y+1}{1} = \frac{z-0}{-1} \).

• Equation of a plane with direction \( \langle a, b, c \rangle \) passing through a point \( (x_0, y_0, z_0) \) is \( a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \).

• Equation of a line with normal direction \( n = \langle a, b, c \rangle \) passing through a point \( (x_0, y_0, z_0) \) is \( \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \).

5. Let \( V, r, h \) be the volume, radius, height. Then \( V(r, h) = \pi r^2 h \). We want to calculate \( \Delta V = V(2.05, 10.20) - V(2, 10) \) using linear approximation.

\[ \Delta V \approx dV = 2\pi rh \, dr + \pi r^2 \, dh. \]

Since \( r = 2, h = 10, dr = 0.05 \) and \( dh = 0.20 \) (top and bottom), we get \( dV = 2.80\pi \).

6. (Look at the subsection in pp.964 up to equality 18). Since \( C \) lies on \( S \), any point \( (x(t), y(t), z(t)) \) must satisfy the equation of \( S \), that is, \( F(x(t), y(t), z(t)) = k \).

Now take \( \frac{d}{dt} \) of both sides. By the Chain Rule (for the left side), we see \( \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} = 0 \), which is, \( \nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0 \).

Therefore, \( \nabla F(x_0, y_0, z_0) \) is perpendicular to \( \mathbf{r}'(t_0) \).

7. Calculate \( \frac{\partial z}{\partial s} \) and \( \frac{\partial z}{\partial t} \) (using chain rule), and multiply them.

\[ \frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \]

\[ \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \]

Therefore,

\[ \frac{\partial z}{\partial s} \cdot \frac{\partial z}{\partial t} = \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \right) \left( \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right) = \left( \frac{\partial f}{\partial x} \right)^2 - \left( \frac{\partial f}{\partial y} \right)^2 = \left( \frac{\partial z}{\partial x} \right)^2 - \left( \frac{\partial z}{\partial y} \right)^2. \]

8. \( (0, 0) \) – saddle point, \( (1, 1) \) – local maximum.

10. (a) Find all local extreme values: Get only one critical point \( (0, 0) \), and \( f(0, 0) = 1 \).

(b) For the boundary \( x^2 + 4y^2 = 1 \), we use the Lagrange multiplier method.

\[ f(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{2\sqrt{2}}) = e^{1/4} \approx 1.284 \text{ (max)}, \quad f(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{2\sqrt{2}}) = e^{-1/4} \approx 0.779 \text{ (min)}. \]