Hints for 14.1 and 14.2

Try all without this hint

§14.1-#22
The only restriction for \( f(x, y, z) \) to be defined is that the argument of \( \ln \) be positive. Thus, the domain is

\[
16 - 4x^2 - 4y^2 - z^2 > 0.
\]

Draw this surface on the \( xyz \)-space.

§14.1-#30
The graph of \( z = \sqrt{4x^2 + y^2} \) is a surface. Try to find the intersection with \( zx \)-plane (Plug in \( y = 0 \)); the intersection with \( zy \)-plane (Plug in \( x = 0 \)). You will get \( z = 2|x|, z = |y| \). After that, look at the level curves \( \sqrt{4x^2 + y^2} = k \) which is \( 4x^2 + y^2 = k^2 \). From these, one can draw an almost complete graph.

§14.1-#32
All the functions in this problem are symmetric in \( x \) and \( y \). To see the cross section with the \( yz \)-plane, plug in \( x = 0 \) to the function. For example, for \( f(x, y) = |x| + |y| \), it becomes \( z = |y| \). On \( yz \)-plane, it’s a \( V \)-shaped broken line.

§14.1-#38
For various \( k \), try to cut the graph by the plane \( z = k \) (it is the plane parallel to the \( xy \)-plane, of height \( k \)). For certain height \( k \), you get 4 points. (For \( k \) bigger than that one, get empty set). As \( k \) gets smaller, you get 4 circles. This circles get bigger until \( k \) becomes 0, when the level curve becomes whole plane with 4 disks punched out. For \( k < 0 \), level curves are empty sets.

§14.1-#44
Draw graph of \( x^3 - y = k \) on the \( xy \)-plane for various values of \( k = \cdots, -2, -1, 0, 1, 2, \cdots \).

§14.1-#66
Draw graph of \( x^2 + 3y^2 + 5z^2 = k^2 \) on the \( xyz \)-space for various values of \( k = \cdots, -2, -1, 0, 1, 2, \cdots \). These are ellipsoids (find their intercepts with the coordinate axes).

§14.2-#8
Try various directions, and you’ll see it is converging. Formally, use polar coordinates. Let

\[
\begin{align*}
x &= 1 + r \cos \theta \\
y &= r \sin \theta
\end{align*}
\]

Then the fraction becomes

\[
\frac{1 + r()}{1 + r()},
\]

where both () are bounded as \( r \to 0 \).

§14.2-#14
One can divide out by \( (x^2 + y^2) \) [why is this possible?] to get a polynomial \( x^2 - y^2 \), which has always has limit (with limit value just \( (x, y) \to (0, 0) \) plugged in).

Reason for [why is this possible?): When you talk about \( \lim_{(x,y)\to(0,0)} f(x,y) \), the value \( f(0,0) \) itself is irrelevant (the value may not even be defined). We only care for the values of \( f(x, y) \) for \( (x, y) \neq (0, 0) \) and \( (x, y) \to (0, 0) \).

§14.2-#32
The only bad point is the zeros of the denominator.

§14.2-#38
For the first function, the only bad point is the zeros of the denominator. At \((0,0)\), the limit of the first function does not exist (show this). The remark above in the box applies here also.

§14.2-#40
Use

\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta.
\end{align*}
\]

After that, you will need l’Hospital’s rule to calculate the limit.