Resistance analysis of infinite networks

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Definition (Electrical resistance network \((G, c)\))

A network \((G, c)\) is a simple, connected graph \(G = \{G^0, G^1\}\) with vertices \(G^0\) and edges \(G^1\).

The edges \(G^1\) are determined by a weight function called conductance:

\[ x \sim y \text{ iff } 0 < c_{xy} < \infty. \]
Definition (Electrical resistance network \((G, c)\))

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The edges \(G^1\) are determined by a weight function called conductance:

\[
x \sim y \text{ iff } 0 < c_{xy} < \infty.
\]

The conductance \(c : G^0 \times G^0 \rightarrow [0, \infty)\) satisfies

\[
c(x) := \sum_{y \sim x} c_{xy} < \infty, \text{ for all } x \in G^0, \text{ and}
\]

\[
c_{xy} = c_{yx} \text{ for all } x, y \in G^0.
\]

Conductance is the reciprocal of the resistance.

What is the natural way to understand \((G, c)\) as a metric space?
Examples of electrical resistance networks

Integer lattice with constant conductances: \((\mathbb{Z}^d, 1)\).

(Homogeneous) trees or other Cayley graphs, with constant conductances or with
\[
c_{xy} = \lambda |x| \wedge |y|, \quad \text{for some } 0 < \lambda < 1.
\]

Relation to fractals: analysis on PCF fractals is developed as a renormalized limiting case of ERNs in [Kig], [Str], etc.
Definition ((Dirichlet) energy of a function $u : G^0 \to \mathbb{R}$)

$\mathcal{E}(u) := \frac{1}{2} \sum_{x, y \in G^0} c_{xy} (u(x) - u(y))^2$, \quad $\text{dom } \mathcal{E} = \{ u : \mathcal{E}(u) < \infty \}$.

$c_{xy} = 0$ unless $x \sim y$; only pairs for which $x \sim y$.

"$\frac{1}{2}$" indicates each edge is counted only once.

For $f : \mathbb{R} \to \mathbb{R}$, the continuous analogue is $\mathcal{E}(f) := \int |f'|^2 \, dx$.

Note: $\ker \mathcal{E} = \{ \text{constant functions} \}$. 
Definition ((Dirichlet) energy of a function $u : G^0 \to \mathbb{R}$)

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$c_{xy} = 0$ unless $x \sim y$; only pairs for which $x \sim y$.

"$\frac{1}{2}$" indicates each edge is counted only once.

Definition (Energy form $\mathcal{E}$ on functions $u, v \in \text{dom } \mathcal{E}$)

$\mathcal{E}(u, v) := \frac{1}{2} \sum_{x,y \in G^0} c_{xy} (u(x) - u(y))(v(x) - v(y))$

(Polarization) $\mathcal{E}(u, v) = \frac{1}{4}[\mathcal{E}(u + v) - \mathcal{E}(u - v)]$.

(Markov property) $\mathcal{E}([u]) \leq \mathcal{E}(u)$, where $[u]$ is any contraction of $u$. 
Definition ((Network) Laplacian \( \Delta \))
A linear difference operator; weighted average of neighbouring values.

\[
(\Delta v)(x) := \sum_{y \sim x} c_{xy} (v(x) - v(y)).
\]

If the operator \( c \) is multiplication by \( c(x) := \sum_{y \sim x} c_{xy} \), then

\[
\Delta = c - T,
\]

where \( T \) is the *transfer operator* (weighted adjacency matrix)

\[
(Tv)(x) := \sum_{y \sim x} c_{xy} v(y).
\]
Definition (Effective resistance)

The **effective resistance** $R(x, y)$ is the voltage drop between $x$ and $y$ when one unit of current is passed from $x$ to $y$.

**Series addition of resistors:** $R = R_1 + R_2$.

**Parallel addition of resistors:** $R = \left(\frac{1}{R_1^{-1} + R_2^{-1}}\right)^{-1}$. 
Definition (Effective resistance)

The effective resistance $R(x, y)$ is the voltage drop between $x$ and $y$ when one unit of current is passed from $x$ to $y$.

Let $I$ be a flow from $x$ to $y$:

$$\text{div } I = A(\delta_x - \delta_y), \text{ for some } A > 0.$$ 

If $I$ is induced by $v$, then $(v(x) - v(y))/A$ is independent of $v$.

$$R(x, y) = (v(x) - v(y))/A.$$ 

Recall Ohm’s law: $V = IR$, or $R = \frac{V}{I}$.
Resistance metric sees the topology of the graph

**Theorem:** Effective resistance $R(x, y)$ is a metric.
For shortest-path metric, $\text{dist}(a, b) = 4 = \text{dist}(x, y)$.

$\text{diff}_{xy} \equiv 1$

Diffusion through the network from $x$ to $y$ is much faster than from $a$ to $b$. To see this, attach the electrodes!


**Theorem:** Effective resistance \( R(x, y) \) is a metric.

For shortest-path metric, \( \text{dist}(a, b) = 4 = \text{dist}(x, y) \).

Points are closer when there are more paths between them:

\[
R(a, b) = 2 > 1 \frac{1}{2} = R(x, y).
\]
Definition (Effective resistance)

The effective resistance \( R(x, y) \) is the voltage drop between \( x \) and \( y \) when one unit of current is passed from \( x \) to \( y \).

\[
R(x, y) = \min \{ v(x) - v(y) : \Delta v = \delta_x - \delta_y \} 
= \min \{ \mathcal{E}(v) : \Delta v = \delta_x - \delta_y \} 
= \min \{ D(I) : \text{div} I = \delta_x - \delta_y \} 
= (\min \{ \mathcal{E}(u) : u(x) = 0, u(y) = 1 \})^{-1} 
= \min \{ \kappa \geq 0 : |v(x) - v(y)|^2 \leq \kappa \mathcal{E}(v) \} 
= \max \{|v(x) - v(y)|^2 : \mathcal{E}(v) \leq 1 \}
\]

The dissipation of a current is \( D(I) := \sum_{e \in G^1} c_{xy}^{-1} I(e)^2 \).

The divergence of a current \( I \) at \( x \in G^0 \) is \( \text{div}(I)(x) := \sum_{y \sim x} I(x, y) \).
\( R(x, y) \) is closely related to \( \Delta \) and the random walk

Let \( X_n \) be a RW started at \( x \), i.e., \( X_0 = x \).

Then the probability of reaching \( b \) before \( a \) is

\[
  u(x) = \frac{v(x)}{R(a, b)} = \mathbb{P}[\tau_b < \tau_a].
\]

Here, \( u, v \) are the extremizers from

\[
  R(x, y) = 1 / \min\{\mathcal{E}(u) : u(x) = 0, u(y) = 1\}
\]

\[
  = \min\{v(y) - v(x) : \Delta v = \delta_x - \delta_y\}
\]

\[
  = \min\{\mathcal{E}(v) : \Delta v = \delta_x - \delta_y\}
\]

\[
  = \min\{\kappa \geq 0 : |v(x) - v(y)|^2 \leq \kappa \mathcal{E}(v)\}
\]

\[
  = \max\{|v(x) - v(y)|^2 : \mathcal{E}(v) \leq 1\}
\]

Again, the RW has \( p(x, y) = \frac{c_{xy}}{c(x)} \), where \( c(x) := \sum_{y \sim x} c_{xy} \).
Extending resistance metric to infinite networks

For a finite subset $H \subseteq G^0$ containing $a, b$, define $R_H(a, b)$ just as before, using any of the six formulas, except now extremizing over $u, v : H \to \mathbb{R}$.

Let $\{G_k\}_{k=1}^{\infty}$ be an exhaustion of $G$:

$G_k \subseteq G_{k+1}$, $G = \bigcup G_k$, each $G_k$ is finite and connected.

Then define $R(x, y) := \lim_{k \to \infty} R_{G_k}(x, y)$. 
Extending resistance metric to infinite networks

For a finite subset $H \subseteq G^0$ containing $a, b$, define $R_H(a, b)$ just as before, using any of the six formulas, except now extremizing over $u, v : H \rightarrow \mathbb{R}$.

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Then define $R(x, y) := \lim_{k \rightarrow \infty} R_{G_k}(x, y)$.

**PROBLEM:** Some of the six formulas still give the right answer, but others don’t. Example:

$$\min \{ \mathcal{E}(v) : \Delta v = \delta_x - \delta_y \} < \lim_{k \rightarrow \infty} \min \{ \mathcal{E}(v_k) : \Delta v_k = \delta_x - \delta_y \text{ on } G_k \}$$
Extending resistance metric to infinite networks

For a finite subset $H \subseteq G^0$ containing $a, b$, define $R_H(a, b)$ just as before, using any of the six formulas, except now extremizing over $u, v : H \rightarrow \mathbb{R}$.

Let $\{G_k\}_{k=1}^\infty$ be an exhaustion of $G$:

$G_k \subseteq G_{k+1}$, $G = \bigcup G_k$, each $G_k$ is finite and connected.

Then define $R^F(x, y) := \lim_{k \to \infty} R_{G_k}(x, y)$. $F$ is for free.
Extending resistance metric to infinite networks

For a finite subset $H \subseteq G^0$ containing $a, b$, define $R_H(a, b)$ just as before, using any of the six formulas, except now extremizing over $u, v : H \to \mathbb{R}$.

Let $\{G_k\}_{k=1}^\infty$ be an exhaustion of $G$:

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Then define $R^F(x, y) := \lim_{k \to \infty} R_{G_k}(x, y)$. $F$ is for free.

For a finite subset $H \subseteq G^0$, define $H^W$ to be the network obtained by identifying ("wiring together") all vertices in $G \setminus H$ to a new vertex called $\infty$ with $c_{x\infty} := \sum_{y \sim x, y \notin H} c_{xy}$.

Then define $R^W(x, y) := \lim_{k \to \infty} R_{G_k^W}(x, y)$. $W$ is for wired.
Resistance analysis of infinite networks

The Hilbert space formalism

Resistance metric on infinite networks

For the exhaustion \( \{G_k\}_{k=1}^{\infty} \):

\[ G_k \subseteq G_{k+1}, \ G = \bigcup G_k, \] each \( G_k \) is finite and connected.
Form $G_k^W$ by identifying all vertices of $G_k^C$ to some new vertex $\infty$. 
This is electrically equivalent to requiring that all neighbours of \( \partial G_k \) have the same potential.

(Hence no current flows through \( G_k^C \).)
Theorem: $R^F, R^W$ are metrics on $(G, c)$ and $R^F(x, y) \geq R^W(x, y)$.

Strict inequality can only happen when $\text{dom} \mathcal{E}$ contains nonconstant harmonic functions.

Lemma: Let $v, h \in \text{dom} \mathcal{E}$. If $v$ is a solution of $\Delta v = \delta_x - \delta_y$ and $h$ is nonconstant and harmonic on $G$, then $\mathcal{E}(v) \neq \mathcal{E}(v + h)$. 
Extending resistance metric to infinite networks

Theorem: \( R^F, R^W \) are metrics on \((G, c)\) and \( R^F(x, y) \geq R^W(x, y) \).

Strict inequality can only happen when \( \text{dom} \mathcal{E} \) contains nonconstant harmonic functions.

Lemma: Let \( v, h \in \text{dom} \mathcal{E} \). If \( v \) is a solution of \( \Delta v = \delta_x - \delta_y \) and \( h \) is nonconstant and harmonic on \( G \), then \( \mathcal{E}(v) \neq \mathcal{E}(v + h) \).

Historically, this is the problem of nonuniqueness of currents.

Given some initial data
\[
\text{div} \ I = \sum_{x \in X} \xi_x \delta_x,
\]
how can you tell when there is a unique current \( I \) with this divergence?
The energy space $\mathcal{H}_\mathcal{E}$

$\text{dom } \mathcal{E} / \mathbb{R}1$ is a Hilbert space

$$\mathcal{H}_\mathcal{E} = \text{dom } \mathcal{E} / \mathbb{R}1, \quad \langle u, v \rangle_\mathcal{E} := \mathcal{E}(u, v).$$

Fix a reference vertex $o \in G^0$, once and for all.
Define $L_x : \text{dom } \mathcal{E} \rightarrow \mathbb{R}$ by $L_x u := u(x) - u(o)$.

$L_x$ is continuous on $\mathcal{H}_\mathcal{E}$, so $L_x u = \langle v_x, u \rangle_\mathcal{E}$ for some $v_x \in \mathcal{H}_\mathcal{E}$.

**Theorem:** $x \mapsto v_x$ is an isometric embedding of $(G, R^F)$ into $\mathcal{H}_\mathcal{E}$:

$$R^F(x, y) = \| v_x - v_y \|_\mathcal{E}^2.$$
The energy space $\mathcal{H}_E$

$\text{dom } \mathcal{E}/\mathbb{R}1$ is a Hilbert space

$$\mathcal{H}_E = \text{dom } \mathcal{E}/\mathbb{R}1, \quad \langle u, v \rangle_\mathcal{E} := \mathcal{E}(u, v).$$

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**Theorem:** $x \mapsto v_x$ is an isometric embedding of $(G, R^F)$ into $\mathcal{H}_E$:

$$R^F(x, y) = \|v_x - v_y\|^2_\mathcal{E}.$$

**Definition**

$v_x$ is a *dipole*. The collection $\{v_x\}_{x \in G^0}$ is the *energy kernel*.

**Theorem:** $\{v_x\}_{x \in G^0}$ is a *reproducing kernel*:

$$\langle v_x, u \rangle_\mathcal{E} = u(x) - u(o) \text{ for all } u \in \mathcal{H}_E.$$
The structure of $\mathcal{H}_\varepsilon$

$\text{dom } \mathcal{E}/\mathbb{R}^1$ is a Hilbert space

$$\mathcal{H}_\varepsilon = \text{dom } \mathcal{E}/\mathbb{R}^1, \quad \langle u, v \rangle_\varepsilon := \mathcal{E}(u, v).$$

**Theorem:** $\mathcal{H}_\varepsilon = \text{Fin} \oplus \mathcal{H}_{\text{arm}}$, where

$\mathcal{H}_{\text{arm}} := \{ h \in \mathcal{H}_\varepsilon : \Delta h(x) = 0, \forall x \in G^0 \}$, and

$\text{Fin} := [\{ f \in \mathcal{H}_\varepsilon : f(x) = k, \text{ for all but finitely many } x \in G^0 \}]_\varepsilon.$
The structure of $\mathcal{H}_\mathcal{E}$

$\text{dom} \mathcal{E}/\mathbb{R}1$ is a Hilbert space

$$\mathcal{H}_\mathcal{E} = \text{dom} \mathcal{E}/\mathbb{R}1, \quad \langle u, v \rangle_\mathcal{E} := \mathcal{E}(u, v).$$

**Theorem:** $\mathcal{H}_\mathcal{E} = \mathcal{F}_\text{in} \oplus \mathcal{H}_\text{arm}$, where

- $\mathcal{H}_\text{arm} := \{ h \in \mathcal{H}_\mathcal{E} : \Delta h(x) = 0, \forall x \in G^0 \}$, and
- $\mathcal{F}_\text{in} := [\{ f \in \mathcal{H}_\mathcal{E} : f(x) = k, \text{ for all but finitely many } x \in G^0 \}]_\mathcal{E}$.

**Theorem (Discrete Gauss-Green Formula)**

$$\langle u, v \rangle_\mathcal{E} = \sum_{G^0} u \Delta v + \sum_{\text{bd } G} u \frac{\partial v}{\partial n}$$

$$\int_U \nabla \varphi \cdot \nabla \psi \, dV = -\int_U \varphi \Delta \psi \, dV + \int_{\partial U} \varphi \frac{\partial}{\partial n} \psi \, dS$$
The structure of $\mathcal{H}_E$

$\text{dom } E / \mathbb{R}1$ is a Hilbert space

$$\mathcal{H}_E = \text{dom } E / \mathbb{R}1, \quad \langle u, v \rangle_E := E(u, v).$$

**Theorem:** $\mathcal{H}_E = \text{Fin} \oplus \text{Harm}$, where

- $\text{Harm} := \{h \in \mathcal{H}_E : \Delta h(x) = 0, \forall x \in G^0\}$, and
- $\text{Fin} := [\{f \in \mathcal{H}_E : f(x) = k, \text{ for all but finitely many } x \in G^0\}]_E$.

**Theorem (Discrete Gauss-Green Formula)**

$$\langle u, v \rangle_E = \sum_{G^0} u \Delta v + \sum_{\text{bd } G} u \frac{\partial v}{\partial n}$$

For $v = f + h$, with $f \in \text{Fin}$, $h \in \text{Harm}$, $E(v) = E(f) + E(h)$

$$\| v \|_E^2 = \sum_{G^0} f \Delta f + \sum_{\text{bd } G} h \frac{\partial h}{\partial n}$$
Theorem (Discrete Gauss-Green)

For \( u, v \in \text{dom } \mathcal{E} \), \( \langle u, v \rangle_{\mathcal{E}} = \sum_{G^0} u \Delta v + \sum_{\text{bd } G} u \frac{\partial v}{\partial n}. \)

Let \( G_k \subseteq G_{k+1}, G = \bigcup G_k \), as before.

\[
\text{bd } G_k := \{ x \in G_k : \exists y \in G_k^c, y \sim x \}
\]

\[
\frac{\partial v}{\partial n}(x) := \sum_{y \in G_k} c_{xy}(v(x) - v(y)), \quad x \in \text{bd } G_k
\]

Think: \( \frac{\partial v}{\partial n}(x) = \Delta|_{G_k}(x). \)
Theorem (Discrete Gauss-Green)

For \( u, v \in \text{dom } \mathcal{E} \),

\[
\langle u, v \rangle_{\mathcal{E}} = \sum_{G^0} u \Delta v + \sum_{\partial G} u \frac{\partial v}{\partial n}.
\]
Theorem (Discrete Gauss-Green)

For \( u, v \in \text{dom } \mathcal{E} \), \( \langle u, v \rangle_{\mathcal{E}} = \sum_{G^0} u \Delta v + \sum_{\text{bd } G} u \frac{\partial v}{\partial n} \).

\[
\sum_{\text{bd } G} u \frac{\partial v}{\partial n} := \lim_{k \to \infty} \sum_{x \in \text{bd } G_k} u(x) \frac{\partial v}{\partial n}(x).
\]
Theorem (Discrete Gauss-Green)

For $u, v \in \text{dom } \mathcal{E}$, $\langle u, v \rangle_\mathcal{E} = \sum_{G^0} u \Delta v + \sum_{\partial G} u \frac{\partial v}{\partial n}$.

Theorem (PJ & EP)

The following are equivalent:

1. $\mathcal{H}_\mathcal{E} = \text{Fin} = [\text{functions of finite support}]_{\mathcal{E}} = [\text{span } \{\delta_x\}]_{\mathcal{E}}$.
2. $\mathcal{H}_{\text{arm}} = 0$.
3. $\sum_{\partial G} u \frac{\partial v}{\partial n} = 0$ for all $u, v \in \mathcal{H}_\mathcal{E}$. (i.e., $\mathcal{E}(u, v) = \langle u, \Delta v \rangle_{\mathcal{E}^2}$.)

Theorem

$\mathcal{H}_\mathcal{E} = \text{Fin} \oplus \mathcal{H}_{\text{arm}}$. 
\[ R^F(x, y) = (v_x - v_y)(x) - (v_x - v_y)(y) \quad (2a) \]
\[ = \mathcal{E}(v_x - v_y) \quad (2b) \]
\[ = \min\{D(I) : \text{div} \, I = \delta_x - \delta_y \text{ and } I = \sum \xi_\gamma \chi_\gamma \} \quad (2c) \]
\[ = (\min\{\mathcal{E}(v) : v(x) = 1, v(y) = 0\})^{-1} + \mathcal{E}(P_{\text{Harm}}(v_x - v_y)) \quad (2d) \]
\[ = \inf\{\kappa \geq 0 : |v(x) - v(y)|^2 \leq \kappa \mathcal{E}(v), \forall v \in \text{dom} \, \mathcal{E} \} \quad (2e) \]
\[ = \sup\{|v(x) - v(y)|^2 : \mathcal{E}(v) \leq 1, \forall v \in \text{dom} \, \mathcal{E} \} \quad (2f) \]

\[ R^W(x, y) = \min\{v(x) - v(y) : \Delta v = \delta_x - \delta_y, v \in \text{dom} \, \mathcal{E} \} \quad (3a) \]
\[ = \min\{\mathcal{E}(v) : \Delta v = \delta_x - \delta_y, v \in \text{dom} \, \mathcal{E} \} \quad (3b) \]
\[ = \min\{D(I) : \text{div} \, I = \delta_x - \delta_y \} \quad (3c) \]
\[ = (\min\{\mathcal{E}(v) : v(x) = 1, v(y) = 0\})^{-1} \quad (3d) \]
\[ = \inf\{\kappa \geq 0 : |v(x) - v(y)|^2 \leq \kappa \mathcal{E}(v), \forall v \in \text{Fin} \} \quad (3e) \]
\[ = \sup\{|v(x) - v(y)|^2 : \mathcal{E}(v) \leq 1, \forall v \in \text{Fin} \} \quad (3f) \]
R^F \text{ vs. } R^W \text{ in terms of boundary conditions on } \triangle

R^F(x, y) = u(x) - u(y) \text{ where } u \text{ is the limit of the solutions to}
\begin{align*}
\Delta u &= \delta_x - \delta_y, \quad \text{on } G_k, \\
\frac{\partial u}{\partial n} &= 0, \quad \text{on } G \setminus G_k,
\end{align*}

R^W(x, y) = u(x) - u(y) \text{ where } u \text{ is the limit of the solution to}
\begin{align*}
\Delta u &= \delta_x - \delta_y, \quad \text{on } G_k, \\
u &= \text{const}, \quad \text{on } G \setminus G_k.
\end{align*}

\begin{align*}
\mathcal{H}E \bigg|_H^F &= \{ u \in \text{dom } E : u(x) - u(y) = 0 \text{ unless } x, y \in H \} \\
\text{and} \quad \mathcal{H}E \bigg|_H^W &= \{ u \in \mathcal{H}E : \text{spt } u \subseteq H \}
\end{align*}

Both are spaces of functions which have no energy outside of \( H \).
Shortcuts through $\infty$

\[
R^F(x, y) = \min\{D(I) : \text{div } I = \delta_x - \delta_y \text{ and } I = \sum \xi_\gamma \chi_\gamma\}
\]

\[
R^W(x, y) = \min\{D(I) : \text{div } I = \delta_x - \delta_y\}
\]

\[
E(v_x) = 1 = R^F(x, o)
\]
Shortcuts through $\infty$

\[ R^F(x, y) = \min \{ D(I) : \text{div} I = \delta_x - \delta_y \text{ and } I = \sum \xi_\gamma \chi_\gamma \} \]

\[ R^W(x, y) = \min \{ D(I) : \text{div} I = \delta_x - \delta_y \} \]
Resistance analysis of infinite networks

The Hilbert space formalism

The discrete Gauss-Green formula

**Shortcuts through \(\infty\)**

\[
R^F(x, y) = \min\{D(I) : \text{div } I = \delta_x - \delta_y \text{ and } I = \sum \xi_\gamma \chi_\gamma\}
\]

\[
R^W(x, y) = \min\{D(I) : \text{div } I = \delta_x - \delta_y\}
\]

![Diagram showing shortcuts](image)

\[
R(x, o) := \min\{\mathcal{E}(u) : \Delta u = \delta_x - \delta_o\}
\]

\[
\mathcal{E}(f_x) = \frac{3}{4} < 1 = \mathcal{E}(v_x)
\]

Note that some current “paths” pass through \(\infty\).
A nontrivial harmonic function in $\text{dom } \mathcal{E}$

Let $c_{xy} = 1$. Then $\mathcal{E}(h) = 1$ and $\lim_{n \to \pm\infty} h(x_n) = \pm 1$.

$h(x) = \pm \left( 1 - \frac{1}{2^{\left| x \right|}} \right)$

Intuition: $\mathcal{Harm} \neq 0$ means the network “grows fast”.

More precisely, $(G, c)$ is an expander: $\inf_{|S| < \infty} \frac{|\text{bd } S|}{|S|} > 0$. 
Theorem (Discrete Gauss-Green formula)  
For $u, v \in \text{dom }\mathcal{E}$, $\langle u, v \rangle_\mathcal{E} = \sum_{G^0} u \Delta v + \sum_{bd G} u \frac{\partial v}{\partial n}$.

Corollary (Boundary sum representation for harmonic functions)  
For $u \in \mathcal{Harm}$, and $h_x = P_{\mathcal{Harm}}\nu_x$,  
$$u(x) = \sum_{bd G} u \frac{\partial h_x}{\partial n} + u(o).$$

Proof. Let $v = h_x$. Then $\langle u, h_x \rangle_\mathcal{E} = u(x) - u(o)$.

Recall:  
$$\|v\|_\mathcal{E}^2 = \sum_{G^0} f \Delta f + \sum_{bd G} h \frac{\partial h}{\partial n}$$
Theorem (Discrete Gauss-Green formula)
For $u, v \in \text{dom } \mathcal{E}$, $\langle u, v \rangle_{\mathcal{E}} = \sum_{G^0} u \Delta v + \sum_{\text{bd } G} u \frac{\partial v}{\partial n}$.

Corollary (Boundary sum representation for harmonic functions)
For $u \in \mathcal{Harm}$, and $h_x = P_{\mathcal{Harm}} v_x$,
$$u(x) = \sum_{\text{bd } G} u \frac{\partial h_x}{\partial n} + u(o).$$

Recall the Poisson kernel $k : \Omega \times \partial \Omega \to \mathbb{R}$ from which
$$u(x) = \int_{\partial \Omega} u(y) k(x, dy), \quad y \in \partial \Omega,$$
for any bounded harmonic function $u$.

Fatou-Primalov: a bounded harmonic function can be extended to the boundary almost everywhere. (So $u(y)$ makes sense a.e.)
\[ u(x) = \sum_{bd \ G} u \frac{\partial h_x}{\partial n} + u(o) \quad \leftrightarrow \quad u(x) = \int_{\partial \Omega} u(y)k(x, dy), \ y \in \partial \Omega. \]

Goals:

- A measure space \( bd \ G \) and a measure \( \mathbb{P} \) on it.
- An extension of \( u, h_x \in \mathcal{H}arm \) to elements \( \xi \in bd \ G \).
- A kernel \( k(x, d\xi) := h_x(\xi)d\mathbb{P}(\xi) \) on \( G^0 \times bd \ G \).
- An integral representation \( u(x) = \int_{bd \ G} u(\xi)k(x, d\xi) + u(o) \).
- A concrete realization of \( \xi \in bd \ G \).
Problem: $\mathcal{H}_\mathcal{E}$ is too small to support $\mathbb{P}$.

**Theorem (Nelson):** If $\mu$ is a $\sigma$-finite measure on a Hilbert space $H$, then $\mu H = 0$. 
Problem: $\mathcal{H}_{\mathcal{E}}$ is too small to support $\mathbb{P}$.

**Theorem (Nelson):** If $\mu$ is a $\sigma$-finite measure on a Hilbert space $H$, then $\mu H = 0$.

**Solution:** construct a Gel’fand triple for $\mathcal{H}_{\mathcal{E}}$.

- $S \subseteq \mathcal{H}_{\mathcal{E}} \subseteq S'$.
- $S$ is dense in $\mathcal{H}_{\mathcal{E}}$ with respect to $\mathcal{E}$.
- $S$ has another, strictly finer, “test function” topology.
- $S'$ is the dual of $S$ with respect to the test function topology.

Think: $S = \{\text{test functions}\}$ and $S' = \{\text{distributions}\}$.

The boundary will be some suitable subspace of $S'$. 
The space of test functions ("of rapid decay")

Definition
Let \( V = \text{span}\{v_x\} \) be the \textit{finite linear combinations} of dipoles.

Let \( \Delta_V \) denote any self-adjoint extension of the (graph) closure of the Laplacian when taken to have this dense domain.
The space of test functions ("of rapid decay")

Definition
Let $V = \text{span}\{v_x\}$ be the \textit{finite linear combinations} of dipoles.

Let $\Delta_V$ denote any self-adjoint extension of the (graph) closure of the Laplacian when taken to have this dense domain.

Definition
Define $S := \text{dom}(\Delta_V^\infty) := \bigcap_{p=1}^\infty \text{dom}(\Delta_V^p)$.

$S$ is a Fréchet space with seminorms $\|u\|_p := \|\Delta_V^p u\|_E$. 
A Gel’fand triple for $\mathcal{H}_\varepsilon$

**Theorem.** $S \subseteq \mathcal{H}_\varepsilon \subseteq S'$ is a Gel’fand triple.

The energy form extends to a pairing on $S \times S'$ defined by

$$\langle u, \xi \rangle_{\mathcal{W}} = \langle \Delta_v^p u, \Delta_v^{-p} \xi \rangle_{\varepsilon} = \lim_{n \to \infty} \xi(E_n u).$$

Note: $\Delta_v^{-p} \xi$ is the $p^{th}$ primitive (“antiderivative”) of $\xi$, nothing to do with the inverse of $\Delta_v$.

$$\xi \in S' \iff |\xi(u)| \leq C\|\Delta_v^p u\|_\varepsilon,$$

so $\varphi(\Delta_v^p u) := \langle u, \xi \rangle$ is continuous on $\text{span}\{\Delta_v^p u : u \in \mathcal{H}_\varepsilon\}$. 

A Gel’fand triple for $\mathcal{H}_E$

**Theorem.** $S \subseteq \mathcal{H}_E \subseteq S'$ is a Gel’fand triple.

The energy form extends to a pairing on $S \times S'$ defined by

$$\langle u, \xi \rangle = \langle \Delta^p u, \Delta^{-p} \xi \rangle = \lim_{n \to \infty} \xi(E_n u).$$

Note: $\Delta^{-p} \xi$ is the $p^{th}$ primitive (“antiderivative”) of $\xi$, nothing to do with the inverse of $\Delta_V$.

Note: $E_n u$ is the spectral truncation of $u$.

$$E_n u := \int_{1/n}^{n} E(d\lambda) u.$$  

$E_n u \in S$ because

$$\|\Delta^p V E_n u\|_{\mathcal{E}}^2 \leq \int_{1/n}^{n} \lambda^{2p} \|E(d\lambda) u\|_{\mathcal{E}}^2 \leq n^{2p} \|u\|_{\mathcal{E}}^2.$$  

**Theorem.** $S$ is a dense analytic subspace of $\mathcal{H}_E$ (w.r. $\mathcal{E}$).
A Gel’fand triple for $\mathcal{H}_E$: $S \subseteq \mathcal{H}_E \subseteq S'$

What to do with a Gel’fand triple?

**Minlos:** $\{\text{pos. def. fns on } S\} \leftrightarrow \{\text{Radon prob. meas. on } S'\}$.

$R^F(x, y) = \|v_x - v_y\|^2_{\mathcal{E}}$ is negative semidefinite, so $e^{-\frac{1}{2}||u-v||^2_{\mathcal{E}}}$ is positive definite on $\mathcal{H}_E \times \mathcal{H}_E$. (Bochner)

Now we have $\mathcal{H}_E \subseteq S'$ and $L^2(S', \mathbb{P})$ to work with.
A Gel’fand triple for $\mathcal{H}_E$: $S \subseteq \mathcal{H}_E \subseteq S'$

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Now we have $\mathcal{H}_E \subseteq S'$ and $L^2(S', \mathbb{P})$ to work with.

The Wiener transform $\mathcal{W} : v_x \mapsto \langle v_x, \cdot \rangle_\mathcal{W}$ is an isometric embedding of $\mathcal{H}_E$ into $L^2(S', \mathbb{P})$:

$$\langle u, v \rangle_\mathcal{E} = \int_{S'} \tilde{u} \tilde{v} \, d\mathbb{P}, \quad \tilde{u}(\xi) := \langle u, \xi \rangle_\mathcal{W}.$$
A Gel’fand triple for $\mathcal{H}_E$: $S \subseteq \mathcal{H}_E \subseteq S'$

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$$\langle u, v \rangle_{\mathcal{E}} = \int_{S'} \bar{u}v \, d\mathbb{P}, \quad u(\xi) := \langle u, \xi \rangle_{\mathcal{W}}.$$
Boundary integral representation of $u \in \mathcal{H}arm$

**Theorem.** For $u \in \mathcal{H}arm$ and $h_x = P_{\mathcal{H}arm}v_x$,

$$u(x) = \int_{S'} u(\xi) h_x(\xi) \, d\mathbb{P}(\xi) + u(o).$$

Substitute $u \in \mathcal{H}arm$ and $v = v_x$ into $\langle u, v \rangle_{\mathcal{E}} = \int_{S'} \bar{u}v \, d\mathbb{P}$:

$$\langle v_x, u \rangle_{\mathcal{E}} = \int_{S'} uv_x \, d\mathbb{P} = u(x) - u(o).$$
Boundary integral representation of $u \in \mathcal{Harm}$

**Theorem.** For $u \in \mathcal{Harm}$ and $h_x = P_{\mathcal{Harm}} v_x$, 

$$u(x) = \int_{S'} u(\xi) h_x(\xi) \, d\mathbb{P}(\xi) + u(o).$$

Compare to $u(x) = \sum_{bd \, G} u \frac{\partial h_x}{\partial n} + u(o) = \int_{bd \, G} u(\xi) k(x, d\xi)$.

**Goals:**
- A measure space $bd \, G$ and a measure $\mathbb{P}$ on it.
- An extension of $u, h_x \in \mathcal{Harm}$ to elements $\xi \in bd \, G$.
- A kernel $k(x, d\xi)$ on $G^0 \times bd \, G$.
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- A concrete realization of $\xi \in bd \, G$. 
The kernel $k(x, dP)$

Since $u(x) = \int_{S'} uh_x dP + u(o)$, the obvious choice is $h_x dP$.

A problem: $\int_{S'} k(x, d\xi) = \int_{S'} 1h_x dP = 0$.

One expects $\int_{S'} k(x, d\xi) = 1$. 
The kernel $k(x, dP)$

From the Wiener isometry:

$$L^2(S', P) = \bigoplus_{n=0}^{\infty} \mathcal{H} \otimes^n = \mathbb{C} 1 \oplus \mathcal{H} \oplus \mathcal{H}^2 \oplus \ldots$$

$$\mathcal{H} = \mathcal{W}(\mathcal{H}_\epsilon)$$

$\mathcal{H} \otimes^0 := \mathbb{C} 1$ for a unit “vacuum” vector 1.

$\mathcal{H} \otimes^n$ is the $n$-fold symmetric tensor product of $\mathcal{H}$ with itself.
The kernel \( k(x, d\mathbb{P}) \)

From the Wiener isometry:

\[
L^2(S', \mathbb{P}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^\otimes n = \mathbb{C}1 \oplus \mathcal{H} \oplus \mathcal{H}^2 \oplus \ldots.
\]

\( \mathcal{H} = \mathcal{W}(\mathcal{H}_\varepsilon) \)

\( \mathcal{H}^\otimes 0 := \mathbb{C}1 \) for a unit “vacuum” vector \( 1 \).

\( \mathcal{H}^\otimes n \) is the \( n \)-fold symmetric tensor product of \( \mathcal{H} \) with itself.

\( u \mapsto \langle u, \cdot \rangle \in \mathcal{H}^1, (u, v) \mapsto \langle u, \cdot \rangle \langle v, \cdot \rangle \in \mathcal{H}^2, \) etc.

Observe that \( 1 \) is orthogonal to \( \text{Fin} \) and \( \text{Harm} \), but is not the zero element of \( L^2(S'_G, \mathbb{P}) \).
The kernel \( k(x, d\mathbb{P}) = (1 + h_x) d\mathbb{P} \)

Now \( \int_{S'} k(x, d\mathbb{P}) = \int_{S'} 1 d\mathbb{P} + \int_{S'} h_x d\mathbb{P} = 1. \)

It follows that \( h_x \geq -1 \) \( \mathbb{P} \)-a.e. on \( S' \).

Also, \( k(x, \cdot) \ll \mathbb{P} \) with Radon-Nikodym derivative \( \frac{d\|k_x}{d\mathbb{P}} = 1 + h_x. \)

\( k(x, d\mathbb{P}) = (1 + h_x) d\mathbb{P} \) is supported on \( G^0 \times S' / \mathcal{F} \in \).

Let \( f \in \mathcal{F} \in \). Since \( h_x \) is harmonic,

\[
\int_{S'_G} f k(x, d\mathbb{P}) = \int_{S'_G} (1 + h_x) f d\mathbb{P} \\
= \int_{S'_G} 1 f \mathbb{P} + \int_{S'_G} h_x f d\mathbb{P} \\
= 0 + \langle h_x, f \rangle \mathcal{E} \\
= 0.
\]
The boundary $\text{bd } G$

A path is a sequence of vertices $\gamma = (x_0, x_1, \ldots)$ with $x_i \sim x_{i-1}$.

Define $\gamma \simeq \gamma'$ iff $\lim_{n \to \infty} (h(\gamma_n) - h(\gamma'_n)) = 0$ for every $h \in \mathcal{Harm}$.
The boundary \( \text{bd} \, G \)

A path is a sequence of vertices \( \gamma = (x_0, x_1, \ldots) \) with \( x_i \sim x_{i-1} \).

Define \( \gamma \simeq \gamma' \) iff \( \lim_{n \to \infty} (h(\gamma_n) - h(\gamma'_n)) = 0 \) for every \( h \in \mathcal{Harm} \).

Let \( \beta = [\gamma] \) be such an equivalence class. Define \( \nu_{\gamma} := \lim_{n \to \infty} k(x_n, d\mathbb{P}) \).
The boundary $\text{bd} \ G$

A path is a sequence of vertices $\gamma = (x_0, x_1, \ldots)$ with $x_i \sim x_{i-1}$.

Define $\gamma \simeq \gamma'$ iff $\lim_{n \to \infty} (h(\gamma_n) - h(\gamma'_n)) = 0$ for every $h \in \mathcal{H}_{\text{arm}}$.

Let $\beta = [\gamma]$ be such an equivalence class. Define

$$\nu_\gamma := \lim_{n \to \infty} k(x_n, d\mathbb{P}).$$

Alaoglu’s theorem gives a weak-$\star$ limit, so for any $u \in \mathcal{H}_{\text{arm}},$

$$u(x_n) = \int_{S'/\text{Fin}} u(1 + h_x) \, d\mathbb{P} \xrightarrow{n \to \infty} \int_{S'/\text{Fin}} u \, d\nu_\gamma := u(\beta).$$

So $\text{bd} \ G$ is the set of all equivalence classes of infinite paths in $G$, under this equivalence relation.
The boundary $\text{bd } G$

Compare $\mathds{1}_k(x_n, d\mathbb{P})$ to an approximate identity in Fourier analysis:

$$\int_{S'/\mathcal{F}_{\text{Fin}}} \mathds{1}_k(x_n, d\mathbb{P}) = 1 \text{ for each } n, \text{ and}$$

$$\lim_{n \to \infty} \int_{S'/\mathcal{F}_{\text{Fin}}} \mathds{1}_k(x_n, d\mathbb{P}) \text{ is a Dirac mass (at } \beta).$$
The boundary $\text{bd} \ G$

Compare $1_k(x_n, d\mathbb{P})$ to an approximate identity in Fourier analysis:

$$\int_{S'/\mathcal{F}_{\text{fin}}} 1_k(x_n, d\mathbb{P}) = 1$$

for each $n$, and

$$\lim_{n \to \infty} \int_{S'/\mathcal{F}_{\text{fin}}} 1_k(x_n, d\mathbb{P})$$

is a Dirac mass (at $\beta$).

Intuition: on any finite subset $G_k$, define a probability measure $\mu_x$ on $\text{bd} \ G_k$ by

$$\mu_x(y) := \mathbb{P}_x[X_{\tau_{\text{bd} G_k}} = y], \text{ for all } y \in \text{bd} \ G_k.$$. 
The boundary $\text{bd } G$

Compare $\mathbb{1}(x_n, d\mathbb{P})$ to an approximate identity in Fourier analysis:

$$\int_{S'/\mathcal{F}_\text{in}} \mathbb{1}(x_n, d\mathbb{P}) = 1 \text{ for each } n,$$
$$\lim_{n \to \infty} \int_{S'/\mathcal{F}_\text{in}} \mathbb{1}(x_n, d\mathbb{P}) \text{ is a Dirac mass (at } \beta).$$

Intuition: on any finite subset $G_k$, define a probability measure $\mu_x$ on $\text{bd } G_k$ by

$$\mu_x(y) := \mathbb{P}_x[X_{\tau_{\text{bd } G_k}} = y], \text{ for all } y \in \text{bd } G_k.$$

Consider Brownian motion on a disk with such an exit measure.
Resistance analysis of infinite networks

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Approximating the reproducing kernels on the tree

\[ f_{\mathcal{X}}^{(k)} \]

\[ h_{\mathcal{X}}^{(k)} \]

\[ j = 0, 1, \ldots, k \]

\[ -\frac{2^k - 1}{2^{k+2} - 2}, \frac{2^{k-j} - 1}{2^{k+2} - 2} \]

\[ -\frac{2^j - 1}{2^{k+2} - 2} \]

\[ 1 - \frac{2^k - 1}{2^{k+2} - 2} \]

\[ 1 - \frac{2^{k-1} - 1}{2^{k+2} - 2} \]

\[ j = 0, 1, \ldots, k \]

\[ \mathcal{X} \]

\[ \mathcal{Y} \]
Definition ((Network) Laplacian $\Delta$)
A linear difference operator; weighted average of neighbouring values.

$$(\Delta v)(x) := \sum_{y \sim x} c_{xy} (v(x) - v(y)).$$

If the operator $c$ is multiplication by $c(x) := \sum_{y \sim x} c_{xy}$, then

$$\Delta = c - T,$$

where $T$ is the transfer operator (weighted adjacency matrix).
Definition ((Network) Laplacian $\triangle$)

A linear difference operator; weighted average of neighbouring values.

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If the operator $c$ is multiplication by $c(x) := \sum_{y \sim x} c_{xy}$, then

$$\triangle = c - T,$$

where $T$ is the transfer operator (weighted adjacency matrix).

$\Delta_p = 1 - c^{-1} T$ is the “probabilistic Laplacian”.

$P := c^{-1} T$ gives transitions with probabilities $p(x, y) = \frac{c_{xy}}{c(x)}$. 
Laplacian and random walk

\[ \Delta_p = 1 - c^{-1} T \] is the “Probabilistic Laplacian”.

\[ \mathbf{P} := c^{-1} T \] gives transitions with probabilities \( p(x, y) = \frac{c_{xy}}{c(x)} \).

Let \( \mu \) be a probability measure on \( G^0 \) giving the initial distribution of a random walker.

Then: \( \mu \mathbf{P} \) gives the distribution of the walker after 1 step, and \( \mu \mathbf{P}^n \) gives the distribution after \( n \) steps.

To start a random walk at \( x \in G^0 \), let \( \mu = \delta_x \).
Laplacian and random walk

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Let \( \mu \) be a probability measure on \( G^0 \) giving the initial distribution of a random walker.

Then: \( \mu P \) gives the distribution of the walker after 1 step, and \( \mu P^n \) gives the distribution after \( n \) steps.

To start a random walk at \( x \in G^0 \), let \( \mu = \delta_x \).

Similarly, let \( u \) be a function on \( G^0 \).

Then: \( Pu \) gives the expected value of \( u \) after 1 step, and \( P^n u \) gives the expected value of \( u \) after \( n \) steps.
Laplacian and random walk

Definition (Harmonic function)

\( h \) is \textit{harmonic} on \( F \subseteq G^0 \) iff \( \Delta h(x) = 0 \) for each \( x \in G^0 \).

Definition (Dirichlet problem)

Designate \( B \subseteq G^0 \) as a “boundary”.
Given \( g : B \to \mathbb{R} \), find \( h \) so \( h{\mid}_B = g \) and \( \Delta h(x) = 0 \) for \( x \in G^0 \setminus B \).

Theorem (Doob)

\textit{The solution is given by} \( h(x) = \mathbb{E}(g(X_{\tau_B})) \), \textit{where} \( X_n \) \textit{is the location of the random walker at time} \( n \), \textit{and} \( \tau_B := \min\{n : X_n \in B\} \).

\( \tau_B \) \textit{is called the hitting time of} \( B \).
Trace and Schur complement

For a finite subset $H \subseteq G^0$, write the Laplacian of $G$ in block form with $H$ appearing first:

$$
\Delta = \begin{bmatrix} H & B^T \\ B & D \end{bmatrix}.
$$

The Schur complement is $\Delta_H := A - B^T D^{-1} B$. 
Trace and Schur complement

For a finite subset $H \subseteq G^0$, write the Laplacian of $G$ in block form with $H$ appearing first:

$$\Delta = \begin{bmatrix} A & B^T \\ B & D \end{bmatrix}.$$

The **Schur complement** is $\Delta_H := A - B^T D^{-1} B$.

$\Delta_H$ defines a subnetwork called the **trace** of $G$ to $H$. The trace $H^S$ has the same vertices as $H$ and edges given by

$$c^H_{xy} = c_{xy} + c(x) \mathbb{P}[x \to y | H^c].$$

$\mathbb{P}[x \to y | H^c]$ is the probability that the RW started at $x$ makes it to $y$ without passing through $H$. 
The trace resistance is then defined to be

$$R^S(x, y) := \lim_{k \to \infty} R^G_{k}(x, y),$$

where \(\{G_k\}\) is any exhaustion of \(G\).

**Theorem:** Let \(H^0 = \{x, y\}\) be any two vertices of \(G\). Then the trace resistance can be computed via

$$\Delta_H = \frac{1}{R^S(x, y)} \left[ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] = A - B^T D^{-1} B.$$
Trace and Schur complement

The *trace resistance* is then defined to be

\[ R^S(x, y) := \lim_{k \to \infty} R_{G_k}^S(x, y), \]

where \( \{G_k\} \) is any exhaustion of \( G \).

**Theorem:** Let \( H^0 = \{x, y\} \) be any two vertices of \( G \). Then the trace resistance can be computed via

\[ \Delta_H = \frac{1}{R^S(x, y)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = A - B^T D^{-1} B. \]

**Theorem:** The trace resistance \( R^S(x, y) \) is given by

\[ R^S(x, y) = \frac{1}{c(x) \mathbb{P}[x \to y]}. \]
Trace and Schur complement

The *trace resistance* is then defined to be

\[ R^S(x, y) := \lim_{k \to \infty} R_{G_k}^S(x, y), \]

where \( \{G_k\} \) is any exhaustion of \( G \).

**Theorem:** \( R^S(x, y) = R_{G_k}^S(x, y) \) for all \( k \).
Trace and Schur complement

The \textit{trace resistance} is then defined to be

\[ R^S(x, y) := \lim_{k \to \infty} R^S_{G_k}(x, y), \]

where \( \{G_k\} \) is any exhaustion of \( G \).

\textbf{Theorem:} \( R^S(x, y) = R^S_{G_k}(x, y) \) for all \( k \).

\textbf{Theorem:} \( R^S_{G_k}(x, y) \) decreases monotonically to \( R^S(x, y) \).

Therefore, \( R^F(x, y) = R^S(x, y) \).
Trace and Schur complement

The *trace resistance* is then defined to be

\[ R^S(x, y) := \lim_{k \to \infty} R_{G_k}^S(x, y), \]

where \( \{G_k\} \) is any exhaustion of \( G \).

**Theorem:** \( R_{G_k}(x, y) \) decreases monotonically to \( R^S(x, y) \). Therefore, \( R^F(x, y) = R^S(x, y) \).

Compare:

\[ R^F(x, y) = \frac{1}{c(x) \mathbb{P}[x \to y]} = \frac{1}{c(x) \sum_{\gamma \in \Gamma(x, y)} \mathbb{P}(\gamma)}. \]

\[ R^F(x, y) = \min\{D(I) : \text{div } I = \delta_x - \delta_y \text{ and } I = \sum \xi_{\gamma} \chi_{\gamma}\}. \]

\[ \mathbb{P}(\gamma) = \mathbb{P}(x_0, x_1, \ldots) := \prod_{n=1}^{\infty} p(x_{n-1}, x_n) \]
Why the Schur complement works

\[
\Delta = \begin{bmatrix}
A & B^T \\
B & D
\end{bmatrix}_{H^c}^H = \begin{bmatrix}
c_A - T_A & -T_{B^T} \\
-T_{B^T} & c_B - T_D
\end{bmatrix}.
\]

If \( \ell(G^0) := \{ f : G^0 \to \mathbb{R} \} \), the corresponding mappings are

- \( A : \ell(H) \to \ell(H) \)
- \( B^T : \ell(H^c) \to \ell(H) \)
- \( B : \ell(H) \to \ell(H^c) \)
- \( D : \ell(H^c) \to \ell(H^c) \).
Why the Schur complement works

\[
\Delta = \left[ \begin{array}{cc}
A & B^T \\
B & D
\end{array} \right] = \left[ \begin{array}{cc}
c_A - T_A & -T_{B^T} \\
-T_{B^T} & c_B - T_D
\end{array} \right].
\]

Since \( P = c^{-1} T \), the Schur complement is

\[
\Delta_H = (c_A - T_A) - (-T_{B^T})(c_D - T_D)^{-1}(-T_B)
\]
\[
= c_A - c_A P_A - c_A P_{B^T}(I - P_D)^{-1}c_D^{-1}c_D P_B
\]
\[
= c_A - c_A \left( P_A + P_{B^T} \left( \sum_{n=0}^{\infty} P_D^n \right) P_B \right).
\]

Next, consider an entry of the matrix \( P_A + P_{B^T} \left( \sum_{n=0}^{\infty} P_D^n \right) P_B \).
Why the Schur complement works

The \((x, y)\)th entry of the matrix \(P_A + P_B^T \left( \sum_{n=0}^{\infty} P^n_D \right) P_B:\n
\begin{align*}
P_A(x, y) + \sum_{n=0}^{\infty} \sum_{s,t} P_B^T(x, s) P^D(s, t) P_B(t, y) \\
= \mathbb{P}^{(c)} \left( \{ \gamma \in \Gamma(x, y) \mid_{H^c} : |\gamma| = 1 \} \right) + \sum_{k=2}^{\infty} \mathbb{P}^{(c)} \left( \{ \gamma \in \Gamma(x, y) \mid_{H^c} : |\gamma| = k \} \right) \\
= \mathbb{P}^{(c)} \left( \bigcup_{k=1}^{\infty} \{ \gamma \in \Gamma(x, y) \mid_{H^c} : |\gamma| = k \} \right) \\
= \mathbb{P}^{(c)} \left( \Gamma(x, y) \mid_{H^c} \right) \\
= \mathbb{P}[x \to y] \mid_{H^c}
\end{align*}