Isotopies of 3-manifolds

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An isotopy of a manifold $M$ that starts and ends at the identity diffeomorphism determines an element of $\pi_1(\text{Diff}(M))$. For compact orientable 3-manifolds with at least three nonsimply connected prime summands, or with one $S^2 \times S^1$ summand and one other prime summand with infinite fundamental group, infinitely many integrally linearly independent isotopies are constructed, showing that $\pi_1(\text{Diff}(M))$ is not finitely generated. The proof requires the assumption that the fundamental group of each prime summand with finite fundamental group imbeds as a subgroup of $\text{SO}(4)$ that acts freely on $S^3$ (conjecturally, all 3-manifolds with finite fundamental group satisfy this assumption). On the other hand, if $M$ is the connected sum of two irreducible summands, and for each irreducible summand $P$ of $M$, $\pi_1(\text{Diff}(P))$ is finitely generated, then results of Jahren and Hatcher imply that $\pi_1(\text{Diff}(M))$ is finitely generated.

The isotopies are constructed on submanifolds of $M$ which are homotopy equivalent to a 1-point union of two 2-spheres and some finite number of circles. The integral linear independence is proven by obstruction-theoretic methods.

Key words: 3-manifold, diffeomorphism, isotopy, diffeotopy, obstruction theory, connected sum, reducible, $\text{SO}(4)$, representation, Whitehead product

1 Introduction

For a compact 3-manifold $M$, the mapping class group $\pi_0(\text{Diff}(M))$ has been studied by many authors under various restrictions. For closed 3-manifolds
with finite fundamental group, it has been explicitly calculated in some cases [2,41]. For 3-manifolds which are irreducible, boundary incompressible, and sufficiently large, the seminal work of Waldhausen [48] and later authors [13,25,29,46] relates it to the outer automorphisms of $\pi_1(M)$. More explicit structural information for the mapping class group is obtained in [13,25,29,46]. The case of compressible boundary is examined in [32,36], and results for reducible 3-manifolds appear in [12,15,16,26,27,29,35].

Naturally, one seeks a fuller understanding of $\text{Diff}(M)$ than simply its group of path components, and the homotopy type of $\text{Diff}_0(M)$, the connected component of the identity diffeomorphism, has been examined by several authors. The proof of the Smale Conjecture [9] is fundamental not only for understanding the important case of $M=S^3$ but also for showing that there is no essential difference between diffeomorphisms and PL homeomorphisms. It is a longstanding conjecture (see [47]) that every closed orientable 3-manifold with finite fundamental group is the quotient of the 3-sphere by a free action of a finite subgroup of $\text{SO}(4)$. This brings us to the first part of our conjectural picture for prime manifolds:

**Conjecture 1.1** Let $P$ be a closed 3-manifold with finite fundamental group. Then $P$ admits a metric of constant positive curvature, and $\text{Diff}(P)$ is homotopy equivalent to the (Lie) group of isometries of $P$.

The homotopy equivalence of $\text{Diff}(P)$ with $\text{Isom}(P)$ has been verified for almost all examples containing a one-sided Klein bottle [21,22,37].

For Haken 3-manifolds, Laudenbach [29] showed that $\pi_1(\text{Diff}(P,x_0))$ is trivial. Hatcher [10] generalized Laudenbach’s method to show that $\text{Diff}_0(P,x_0)$ is contractible in the Haken case, and [11] proved that $\text{Diff}(S^2 \times S^1) \simeq \text{O}(2) \times \text{O}(3) \times \Omega \text{O}(3)$. For $P = D^2 \times S^1$, one has the fibration $\text{Diff}_0(D^2 \times S^1 \rightarrow \partial D^2 \times S^1) \rightarrow \text{Diff}_0(D^2 \times S^1) \rightarrow \text{Diff}_0(\partial D^2 \times S^1)$ in which the fiber is contractible by [10] so $\text{Diff}_0(D^2 \times S^1) \simeq \text{Diff}_0(\partial D^2 \times S^1) \simeq S^1 \times S^1$. This brings us to our second conjecture.

**Conjecture 1.2** Let $P$ be a compact orientable prime 3-manifold with infinite fundamental group. If $P = S^2 \times S^1$, then $\text{Diff}(P) \simeq \text{O}(2) \times \text{O}(3) \times \Omega \text{O}(3)$. If $P = D^2 \times S^1$, then $\text{Diff}_0(P) \simeq S^1 \times S^1$. In all other cases, $\text{Diff}_0(P) \simeq (S^1)^k$ where $k$ is the rank of the center of $\pi_1(P)$.

Since the center of a 3-manifold fundamental group must be finitely generated [38], conjectures 1.1 and 1.2 together would imply that $\text{Diff}_0(P)$ has the homotopy type of a finite complex whenever $P$ is a compact orientable prime 3-manifold.

When $M$ is reducible, $\text{Diff}(M)$ has a much more elaborate homotopy type than in the (known) prime cases. Hendriks and Laudenbach [17], refining and
filling gaps in work of Cesar de Sà and Rourke [3], constructed a configuration space whose loop space appears (speaking approximately, in order to avoid technicalities) as a direct factor up to homotopy of $\text{Diff}(M)$. Direct examination of the configuration space has not yielded many further results (although some information was obtained from it in [18]). However, it gives very useful qualitative information about $\text{Diff}(M)$: because the configuration space is, roughly speaking, the space of possible positions for the summands in a manifold diffeomorphic to $M$, it suggests that the homotopy type of $\text{Diff}(M)$ is in large part determined by phenomena that arise from “sliding” irreducible summands around loops in $M$. Thus, it sets a direction for further examination of the homotopy type of $\text{Diff}(M)$.

Below we will explain how these sliding phenomena lead to isotopies on 3-manifolds, and how we will use them in this paper, but first we will give our main results.

**Main Theorem** Let $M$ be a compact orientable 3-manifold which has no 2-sphere boundary components.

(a) Suppose $M$ is prime or is the connected sum of two irreducible summands, and that for each irreducible summand $P$ of $M$, $\pi_1(\text{Diff}(P))$ is finitely generated. Then $\pi_1(\text{Diff}(M))$ is finitely generated.

(b) Suppose $M$ has at least three nonsimply connected prime summands, or is the connected sum of $S^2 \times S^1$ with a prime summand with infinite fundamental group. Assume that for any irreducible summand with finite fundamental group, the fundamental group is isomorphic to a subgroup of $\text{SO}(4)$ which acts freely on $S^3$. Then $\pi_1(\text{Diff}(M))$ has infinite rank as an abelian group.

Note that in part (b), the condition assumed on the summands with finite fundamental group is implied by conjecture 1.1.

The case of manifolds with 2-sphere boundary components is addressed by the following result. In its statement, $\hat{N}$ denotes the manifold obtained from $N$ by filling in all 2-sphere boundary components with 3-balls, and $\mathcal{P}(N)$ denotes the manifold obtained by replacing all homotopy 3-sphere prime summands with $S^3$ summands.

**Theorem 10.1** Let $N$ be a 3-manifold which is compact and orientable. Then $\pi_1(\text{Diff}(N \# D^3))$ is finitely generated if and only if one of the following three holds.

(i) $\pi_1(N)$ is finite and $\pi_1(\text{Diff}(\hat{N}))$ is finitely generated.

(ii) $\mathcal{P}(N)$ is prime and $\pi_1(\text{Diff}(N))$ is finitely generated.

(iii) $N \approx \mathbb{R}P^3 \# \mathbb{R}P^3$ and $\pi_1(\text{Diff}(N))$ is finitely generated.
According to [8], conjecture 1.1 holds for the case of $\mathbb{R}P^3$, so $\text{Diff}(\mathbb{R}P^3) \simeq \text{Isom}(\mathbb{R}P^3) = SO(4)/(\pm 1)$, which by part (a) of the Main Theorem implies that $\pi_1(\text{Diff}(\mathbb{R}P^3 \# \mathbb{R}P^3))$ is finitely generated. So if conjectures 1.1 and 1.2 are true in general, theorem 10.1 becomes simply: $\pi_1(\text{Diff}(N \# D^3))$ is finitely generated if and only if either $\pi_1(N)$ is finite, or $N$ is prime, or $N = \mathbb{R}P^3 \# \mathbb{R}P^3$.

The cases left unresolved by the Main Theorem and theorem 10.1 are the manifolds of the form $S^2 \times S^1 \# P$ where $\pi_1(P)$ is finite. For these, we obtain the following partial information in section 8.

**Theorem 8.3** Let $P$ be a closed 3-manifold with finite fundamental group of order $n$. Then $\pi_1(\text{Diff}(S^2 \times S^1 \# P))$ contains a free abelian subgroup of rank $n - 1$.

We will now summarize the proof of part (b) of the Main Theorem, which occupies almost all of sections 2 through 8 of this paper. A special case of our construction of isotopies appeared in [31] where it was used to prove that if $M$ is a connected sum of at least three closed aspherical 3-manifolds, then $\pi_1(\text{Diff}(M))$ is not finitely generated. The idea was to construct a model isotopy on the 3-manifold $Y_0$ shown in figure 1 below, which is obtained by joining two $S^2 \times I$'s using a 1-handle. Assuming that $M$ has at least 3 summands, $Y_0$ can be imbedded into $M$ in infinitely many homotopically distinct ways, by mapping the $S^2 \times I$'s to neighborhoods of two connected-sum 2-spheres and running the 1-handle around arbitrary loops in a third summand. The model isotopy fixes $\partial Y_0$, so extends using the identity diffeomorphism to an isotopy of $M$. An obstruction-theoretic argument was used to show that a certain infinite collection of these isotopies generates an infinitely generated subgroup of $\pi_1(\text{Diff}(M))$. The cohomological calculations in [31] required the assumption of asphericity on the summands.

In the present paper, the approach of [31] is refined. In section 3 a sequence of 3-manifolds $Y_n$ is constructed, and in section 4 a general “sliding” method of constructing isotopies motivated by the configuration space of [17] is used to construct model isotopies on the $Y_n$. These manifolds are imbedded into $M$ in certain ways, according to the number of $S^2 \times S^1$ summands, and again the model isotopies extend using the identity diffeomorphism to the rest of $M$ to give loops representing elements in $\pi_1(\text{Diff}(M))$. An obstruction-theoretic method, detailed in section 5, can be used to detect that these elements are linearly independent in $\pi_1(\text{Diff}(M))$. This method requires a good understanding of the obstruction theory of the model isotopies on the $Y_n$; the technical work to determine the necessary difference obstructions is carried out in section 4. The work on $M$ breaks into three cases according to the number of $S^2 \times S^1$ summands of $M$, and these are pursued in sections 6, 7, and 8. Proving the linear independence of the difference obstructions involves the use of some rather interesting representations of $\pi_2(M)$ (and the symmetric product
\( S(\pi_2(M)) \), defined in section 5).

Since our method of proving linear independence of the isotopies is completely homotopy-theoretic, it works just as well for spaces of homotopy equivalences. Simply replace \( \text{Diff}(M \text{ rel } x_0) \) and \( \text{Diff}(M) \) respectively by any of

1. \( \text{Equiv}(M \text{ rel } (\partial M \cup x_0)) \) and \( \text{Equiv}(M \text{ rel } \partial M) \)
2. \( \text{Equiv}(M, \partial M \text{ rel } x_0) \) and \( \text{Equiv}(M, \partial M) \)
3. \( \text{Equiv}(M \text{ rel } x_0) \) and \( \text{Equiv}(M) \)

in the main diagram in section 5. The proof of part (b) of the Main Theorem then applies without essential change to show infinite generation of the fundamental groups of these spaces of homotopy equivalences for the manifolds in part (b). The proof of theorem 8.3 also applies without change.

In the proof of part (b) of the Main Theorem and theorem 8.3, the assumption that \( M \) is orientable is used only when writing down \( \mathbb{Z}\pi_1(M) \)-module presentations of \( \pi_2(M) \). Similar infinite generation results should be possible by adapting our approach to the more complicated presentations that occur in the presence of two-sided projective planes or nonorientable \( S^2 \)-bundles over \( S^1 \).

Part (a) of the Main Theorem follows from a stronger result given as theorem 9.2; this result is briefly mentioned in [11], but we fill in some details that to us seem nontrivial. Section 10 is a proof of theorem 10.1, based on different methods from those of the Main Theorem.

Recently, information about the homotopy type of \( \text{Diff}(M) \) has been applied to the theory of quantum gravity. Certain physical configuration spaces can be realized as the quotient space of a principal \( \text{Diff}_1(M, x_0) \)-bundle with contractible total space, where \( \text{Diff}_1(M, x_0) \) denotes the subspace of \( \text{Diff}(M, x_0) \) that induce the identity on the tangent space to \( M \) at \( x_0 \). (This group is homotopy equivalent to \( \text{Diff}(M \# D^3 \text{ rel } \partial D^3) \).) Consequently the loop space of the configuration space is weakly homotopy equivalent to \( \text{Diff}_1(M, x_0) \). Physical significance of \( \pi_0(\text{Diff}(M)) \) for quantum gravity was first pointed out in [5]. See also [1,7,20,42,50]. The significance of some higher homotopy groups of \( \text{Diff}(M) \) is examined in [6].

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2 Lemmas

In this section we collect some miscellaneous lemmas which will be needed in later sections. The first is a version of a homotopy-theoretic fact which may be well-known to algebraic topologists, but unfamiliar to some geometric topologists. Since it does not seem to be an immediate consequence of standard results about Whitehead products, we include a direct proof here. For simplicity we restrict to the case needed for 3-dimensional manifolds, although the proof works just as well in an $n$-dimensional version. For definitions and basic facts about Whitehead products, one can consult standard texts on algebraic topology, such as [49].

For convenience, $M$ is assumed throughout this section to be a manifold without boundary. In the applications of these lemmas in later sections, the manifolds sometimes have boundary, but the lemmas can always be used by taking the $M$ in the lemmas to be the interior of the manifold in the application. Alternatively, one may replace $M$ by int$(M)$ throughout this section.

Suppose that $W$ is a compact submanifold of $M$, and $V$ is a submanifold of $W$. By a theorem of Palais [40], the map Imb$(W, M) \rightarrow$ Imb$(V, M)$ defined by restriction of (smooth) imbeddings is a fibration. In the case when $V$ is a single point $x_0$, Imb$(V, M)$ is identified with $M$, and the fibration is a map Imb$(W, M) \rightarrow M$ with fiber the space Imb$((W, x_0), (M, x_0))$ of basepoint-preserving imbeddings. This appears in the first lemma, where $W$ is a 2-sphere $S$.

Fix a map $\sigma$: $(I^2, \partial I^2) \rightarrow (S, x_0)$ which generates $\pi_2(S, x_0)$. We will use the difference obstruction $d_{q+2}: \pi_q(\text{Imb}((S, x_0), (M, x_0))) \rightarrow \pi_{q+2}(M)$, defined as follows. Fix a relative CW-complex structure on $(I^q \times S, I^q \times x_0)$ having one $(q+2)$-cell $e: I \times I^2 \rightarrow I^q \times S$ which maps each $\{x\} \times I^2$ to $\{x\} \times S$ by $\sigma$. If $w: I^q \times S \rightarrow M$ is a $q$-parameter family of basepoint-preserving imbeddings, for which $w|_{\{x\} \times S}$ is the inclusion whenever $x \in \partial I^q$, then $d_{q+2}(w): S^{q+2} = I^{q+2} \cup_{\partial I^{q+2}} I^{q+2} \rightarrow M$ equals $je$ on the top hemisphere and equals $we$ on the bottom hemisphere, where $j$ is the family that is the inclusion map for each parameter. Composing $d_{q+2}$ with the boundary homomorphism $\partial$ from the long exact sequence for the fibration Imb$((S, x_0), (M, x_0)) \rightarrow \text{Imb}(S, M) \rightarrow M$, we obtain $d_{q+2}\partial: \pi_{q+1}(M) \rightarrow \pi_{q}(\text{Imb}((S, x_0), (M, x_0))) \rightarrow \pi_{q+2}(M)$.

**Lemma 2.1** $d_{q+2}\partial(\tau) = [\sigma, \tau]$, where $[a, b]$ denotes Whitehead product.

**PROOF.** First we describe $\partial: \pi_{q+1}(M) \rightarrow \pi_{q}(\text{Imb}((S, x_0), (M, x_0)))$. Let $\tau \in \pi_{q+1}(M, x_0)$. Regard $I^{q+1}$ as $I^q \times I$, and $\tau$ as a map from $I^{q+1}$ to $M$ which sends $\partial I^{q+1}$ to $x_0$. We will construct a $(q+1)$-parameter family of imbeddings from $S$ to $M$, as a map from $I^{q+1} \times S$ to $M$. For each parameter
in $I^q \times \{1\} \cup \partial I^q \times I$, take the inclusion map. On $I^{q+1} \times x_0$, use a representative of $\tau$ to define the map. Since $\text{Imb}(S, M) \to M$ is a fibration, the homotopy extension property yields a map $W: I^{q+1} \times S \to M$ such that for each $x \in I^{q+1}$, the restriction of $W$ to $\{x\} \times S$ is an imbedding. The restriction $W_1$ of $W$ to $I^q \times \{0\} \times S$ represents the element $\partial(\langle \tau \rangle) \in \pi_q(\text{Imb}((S, x_0), (M, x_0)))$.

Let $W_t = W|_{I^q \times \{t\} \times S}: I^q \times S \to M$. Regard $S^{q+2}$ as $\partial(I^{q+2} \times I)$. The difference element is represented by a map $d: S^{q+2} \to M$ which equals $je$ on $I^{q+2} \times \{1\}$, equals $W_1e$ on $I^{q+2} \times \{0\}$, and equals the restriction of $je$ on $\partial I^{q+2} \times \{t\}$ for each $t \in I$. Now $W_te$ defines a homotopy from $W_1e$ to $W_0e = je$ relative to $\partial I^q \times I^2$. Using the homotopy extension property, we may change $d$ by homotopy to a map $d'$ which equals $je(x, s)$ for each $(x, s, t)$ in $\partial I^q \times I^2 \times I \cup I^q \times I^2 \times \partial I$. During the homotopy from $d$ to $d'$, the restriction to $I^q \times \{s\} \times \{0\}$ sweeps through a map representing $\tau$ for each $s \in \partial I^2$. So for each such $s$, the restriction of $d'$ to $I^q \times \{s\} \times I$ represents $\tau$. Now, regard $S^{q+2}$ as $\partial(I^q \times I^2 \times I) = \partial(I^{q+1} \times I^2)$, where the $I$ and $I^2$ factors from before have now been interchanged. On $\partial I^{q+1} \times I^2$, $d'$ is projection to $I^2$ followed by $\sigma$, while on $I^{q+1} \times \partial I^2$, $d'$ is projection to $I^{q+1}$ followed by a map representing $\tau$. Therefore $d'$ represents $[\sigma, \tau]$. □

To prepare for the next lemma, we must discuss the construction known as a modification in a ball, which will be used at several points later in the paper, especially in section 4. Let $D$ be an imbedded $n$-cell in an $n$-manifold $M$ ($n \geq 2$), and fix a point $x$ in $\partial D$ as the basepoint of $M$. Let $\eta$ be an element of $\pi_n(M, x)$ which can be represented by a map which is not surjective (or equivalently, for triangulated $M$, by a map into the $(n-1)$-skeleton of $M$).

A modification in the ball $D$ by $\eta$ is a continuous map $f: M \to M$ whose restriction to $M - D$ is the identity and whose difference element in $\pi_n(M, x)$, represented by the map $d: S^n = D^n \cup \partial D^n \to M$ with $d|_{\partial D^n} = f|_D$ and $d|_{D^n} = 1_D$, equals $\eta$. Modification in $D$ by $-\eta$ is a homotopy inverse for $f$ (this uses the fact that $\eta$ can be represented by a map which is not surjective), so $f$ is a homotopy equivalence. Now suppose that $D'$ is another ball and $x' \in \partial D'$, and let $j_i$ be an isotopy of $M$ starting at the identity and moving $x$ along a path $\alpha$ from $x$ to $x'$, and suppose that $j_1(D) = D'$. Then each map $j_i fj_i^{-1}$ is a modification in the ball $j_i(D)$ by the element in $\pi_n(M, j_i(x))$ determined by $\eta$ using the portion which is the portion of $\alpha$ running from $j_i(x)$ to $x$. In particular, if $\alpha$ is a closed loop and $j_1(D) = D$, then $j_1 fj_1^{-1}$ is a modification in $D$ by the result of acting on $\eta$ by the reverse of $\alpha$. We refer to such a homotopy as sliding the modification along $\alpha$.

The next lemma is a variation of lemma 2.1. To set notation, let $M$ be a 3-manifold, and let $W$ be an imbedded copy of $S^2 \times I$ in $M$. Fix a basepoint $x_0$ in $S^2$, and let $z: I \to M$ be the imbedding defined by $z(s) = (x_0, s) \in W$. Identify $S^2$ with $S^2 \times \{0\}$ and $x_0$ with $\{x_0\} \times \{0\}$. Fix a map $\tau: (I^2, \partial I^2) \to (M, x_0)$, and define $\tau': I \times I \to M$ as follows:
1) If \((s,t) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times \left[0, \frac{1}{2}\right]\), then \(\tau'(s,t) = \tau\left(2\left(s - \frac{1}{4}\right), 2t\right)\).
2) If \((s,t) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times \left[\frac{1}{2}, 1\right]\), then \(\tau'(s,t) = z(2t - 1)\).
3) If \((s,t) \notin \left[\frac{1}{4}, \frac{3}{4}\right] \times [0, 1]\), let \(M = 4|s - \frac{1}{2}| - 1\). Then

\[
\tau'(s,t) = \begin{cases} 
  x_0, & 0 \leq t \leq \frac{1-M}{2} \\
  z\left(\frac{2t+M-1}{M+1}\right), & \frac{1-M}{2} \leq t \leq 1.
\end{cases}
\]

On the square \(\left[\frac{1}{4}, \frac{3}{4}\right] \times \left[0, \frac{1}{2}\right]\), \(\tau'\) looks like \(\tau\). The triangles with vertex sets \(\{(0,0), \left(\frac{1}{4}, 0\right), \left(\frac{1}{2}, \frac{1}{2}\right)\}\) and \(\{(\frac{3}{4}, 0), (1,0), \left(\frac{1}{2}, \frac{1}{2}\right)\}\) are mapped to \(x_0\), and each maximal vertical segment (i.e. each segment with fixed value of \(s\)) in the closure of the complement of the union of the square and the two triangles is mapped by a linear reparameterization of \(z\).

Define \(F: M \times I \to M\) as follows: for \(x \in M - W\) let \(F(x,t) = x\), for \((y,t) \in W \times \{0\} \cup \partial W \times I\) let \(F(y,t) = y\), and for any point \((z(s), t) \in z(I) \times I\) let \(F(z(s), t) = \tau'(s,t)\). Use the homotopy extension property to extend \(F\) over all of \(M \times I\), and let \(F_1 = F|_{M \times \{1\}}: M \to M\).

**Lemma 2.2** (Whitehead Product Lemma) The map \(F_1\) is homotopic, relative to \((z(I) \cup M - W)\), to a modification in a ball by the Whitehead product \([\sigma, \tau]\).

**Proof.** A relative cell-complex structure on \((W, z(I))\) is given by a level-preserving map \(c: I^3 = I^2 \times I \to S^2 \times I\) defined by \(c(x, s) = (\sigma(x), s) \in S^2 \times I\). Each vertical arc \(\{x\} \times I \subset \partial I^2 \times I\) maps homeomorphically onto \(z(I)\), and \(c|_{\text{int}(I^2) \times I}: \text{int}(I^2) \times I \to (S^2 - x_0) \times I\) is a homeomorphism. Let \(C = c \circ \text{id}: I^3 \times I \to W \times I\) and let \(C_t = C|_{I^2 \times \{t\}}: I^3 \to M\) and \(F_t = F|_{M \times \{t\}}: M \to M\). The difference \(d_3(F_1)\) on the 3-cell may be represented as a map \(d: \partial(I^3 \times I) \to M\) such that \(d|_{I^3 \times \{0\}} = F_1 C_1, d|_{I^3 \times \{1\}} = F_0 C_0,\) and \(d|_{\partial I^3 \times \{t\}} = (F_0 C_0)|_{\partial I^3}\) for \(t \in I\). To prove the lemma, it suffices to show that \(d\) represents \([\sigma, \tau]\).

Now \(F_t C_t\) defines a homotopy from \(F_1 C_1\) to \(F_0 C_0\), relative to \(I^2 \times \partial I\). Using the homotopy extension property, we may change \(d\) by homotopy so that for \((x, s, t) \in I^2 \times I \times \partial I^2 \cup I^2 \times \partial I\times I\), \(d(x, s, t) = (F_0 \circ C_0)(x, s) = (\sigma(x), s)\). For any vertical arc \(z'\) in \(\partial I^2 \times I \subset I^2\), \(F_t C_t(z') = F_t(z(I)) = \tau'(I \times \{t\})\), and therefore we may assume \(d(z' \times \{t\}) = \tau'(I \times \{t\})\). There is a homotopy \(H: \tau' \simeq \tau\), relative to \(I \times \{0\}\), so that the trace of any point on \(I \times \{1\}\) is \(\tau\) where \(\tau(s) = z(1 - s)\). Using this homotopy on each \(z' \times I\) we may further deform \(d\) relative to \(I^2 \times \{0\} \times I\), so that for \((x, s, t) \in I^2 \times \partial I^2 \cup \partial I \times I^2 \times \{0\} \times I \cup I^2 \times \{1\} \times I\), \(d(x, s, t) = \sigma(x)\) and \(d(z' \times \{t\}) = \tau(I \times \{t\})\). Note that \(d|_{z' \times I}\) represents the
element \( \tau \) in \( \pi_2(M, x_0) \). Decomposing \( \partial(I^3 \times I) \) into two solid tori

\[
(I^3 \times \{0\} \cup I^3 \times \{1\} \cup I^2 \times \{0\} \times I \cup I^2 \times \{1\} \times I) \cup (\partial I^2 \times I \times I)
\]

we see that \( d \) represents \([\sigma, \tau] \). □

We will need a special fact about \( \text{SO}(4) \). First recall that two representations \( \phi: G \to \text{GL}(V) \) and \( \phi': H \to \text{GL}(W) \) of groups determine a tensor product representation \( \phi \otimes \phi': G \times H \to \text{GL}(V \otimes W) \), such that the traces satisfy

\[
\chi_{\phi \otimes \phi'}((g, h)) = \chi_\phi(g) \chi_{\phi'}(h).
\]

In particular, \( \text{SU}(2) \) is \( \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid |a^2| + |b^2| = 1 \right\} \) and is a subgroup of \( \text{GL}(2, \mathbb{C}) \), so there is a tensor product representation \( \rho: \text{SU}(2) \times \text{SU}(2) \to \text{GL}(\mathbb{C}^2 \otimes \mathbb{C}^2) = \text{GL}(4, \mathbb{C}) \). One verifies by computation that \( \rho^{-1}(\pm I) = \{(I, I), (I, -I), (-I, I), (-I, -I)\} \). The image of \( \rho \) is compact and connected, so changing \( \rho \) by conjugation we may assume its image lies in the maximal compact connected subgroup \( \text{SO}(4, \mathbb{R}) \) of \( \text{GL}(4, \mathbb{C}) \).

Since \( \text{SU}(2) \times \text{SU}(2) \) and \( \text{SO}(4, \mathbb{R}) \) both have dimension 6, \( \rho \) is surjective. Let \( \tau: \text{SO}(4) \to \text{SO}(4) \) be the automorphism induced on \( \text{SO}(4) = \rho(\text{SU}(2) \times \text{SU}(2)) \) by interchange of the factors in \( \text{SU}(2) \times \text{SU}(2) \).

**Lemma 2.3** Let \( A \) and \( B \) be elements of \( \text{SO}(4) \), neither of which is \( \pm I \). Then either \( AXBX^{-1} \) or \( AX\tau(B)X^{-1} \) has nonconstant trace as \( X \) varies over \( \text{SO}(4) \).

**PROOF.** We are grateful to Mark Reeder for help with the following argument. Assume that the trace of \( AXBX^{-1} \) is constant. Choose elements \( (A_1, A_2), (X_1, X_2), \) and \( (B_1, B_2) \) of \( \text{SU}(2) \times \text{SU}(2) \) that map to \( A, X, \) and \( B \) respectively under \( \rho \). From above, we know that the traces satisfy

\[
\chi_{\rho}((A_1, A_2)(X_1, X_2)(B_1, B_2)(X_1^{-1}, X_2^{-1})) = \chi(A_1X_1B_1X_1^{-1})\chi(A_2X_2B_2X_2^{-1}).
\]

Since \( X_1 \) may be varied while \( X_2 \) is held fixed, \( \chi(A_1X_1B_1X_1^{-1}) \) is constant as \( X_1 \) varies, and similarly \( \chi(A_2X_2B_2X_2^{-1}) \) is constant as \( X_2 \) varies. By conjugating in \( \text{SU}(2) \) we may assume that \( B_1 \) has the form \( \begin{pmatrix} c & 0 \\ 0 & \tau \end{pmatrix} \). A calculation
shows that the trace of
\[
\begin{pmatrix}
a & b \\
-\bar{b} & \bar{a}
\end{pmatrix}
\begin{pmatrix}
x & y \\
-\bar{y} & \bar{x}
\end{pmatrix}
\begin{pmatrix}
c & 0 \\
0 & \bar{c}
\end{pmatrix}
\begin{pmatrix}
x - y \\
y & x
\end{pmatrix}
\]
is \(2|x|^2\Re(ac - a\bar{c}) + 2\Re(a\bar{c}) + 4\Im(c)(\Im(b\bar{xy}) - |x|^2\Im(a)).\) Using the identity \(\Re(ac - a\bar{c}) = -2\Im(a)\Im(c),\) this becomes \(2\Re(a\bar{c}) + 4\Im(c)(\Im(b\bar{xy}) - |x|^2\Im(a)).\) Holding \(x\) fixed and varying \(y\) shows that either \(b = 0,\) in which case \(A_1 = \pm I,\) or \(c\) is real, in which case \(B_1 = \pm I.\)

The corresponding calculation with \(\chi(A_2X_2B_2X_2^{-1})\) shows that either \(A_2 = \pm I,\) or \(B_2 = \pm I.\) Since \(A \neq \pm I,\) we cannot have both \(A_1\) and \(A_2\) equal to \(\pm I,\) and similarly for \(B_1\) and \(B_2,\) so without loss of generality we may assume that \(A_1 \neq \pm I\) and \(B_2 \neq \pm I.\) But then, our calculation shows that the trace of \(A_1X_1B_2X_1^{-1}\) is nonconstant, and hence the trace of \(AX\tau(B)X^{-1}\) is nonconstant.

\[\square\]

3 The manifolds \(Y_n\) and the Duality Theorem

We will use certain 3-manifolds with boundary called \(Y_n,\) where \(n\) is a nonnegative integer. The case when \(n=0\) is important, and all of our statements about \(Y_n\) will hold in this case. The manifolds \(Y_0\) and \(Y_2\) are illustrated in figure 1. To construct \(Y_n,\) let \(S_+\) and \(S_-\) be two 2-spheres, and form \(S_+ \times I\) and \(S_- \times I\) where \(I = [0, 1].\) Choose an imbedded 2-disc \(C_0 \subseteq S_+ \times \{1\},\) disjoint imbedded 2-discs \(D_0, C_2, D_2, C_4, \ldots, C_{2n}, D_{2n}\) in \(S_- \times \{1\},\) and disjoint imbedded 2-discs \(C_1, D_1, C_3, \ldots, C_{2n-1}, D_{2n-1}\) in \(S_- \times \{0\}.\) Attach \(2n + 1\) 1-handles connecting \(C_i\) to \(D_i\) for \(0 \leq i \leq 2n.\)

Clearly \(Y_n\) is homotopy equivalent to a 1-point union of two 2-spheres and \(2n\) circles. Thus \(\pi_1(Y_n)\) is free on \(2n\) generators.

**Lemma 3.1** \(\pi_2(Y_n)\) and \(\pi_3(Y_n)\) are free \(\mathbb{Z}\pi_1(Y_n)\)-modules.

**Proof.** Denote \(\pi_1(Y_n)\) by \(\Gamma.\) The universal cover of \(Y_n\) is homotopy equiv-
alent to a 1-point union $\tilde{Y}$ of 2-spheres. Under the action of $\Gamma$, the set of these spheres has two orbits, each in one-to-one correspondence with the elements of $\Gamma$ with action given by group multiplication. Choose 2-spheres $S_1$ and $T_1$, one in each orbit, and label all the other 2-spheres as $S_g$ and $T_g$ where $S_g = gS_1$ and $T_g = gT_1$. According to Hilton [19], $\pi_2(Y_n)$ is a free $\mathbb{Z}$-module on the generators of $\pi_2(S_g)$ and $\pi_2(T_g)$ as these range over all 2-spheres, so $\pi_2(Y_n)$ is a free $\mathbb{Z}\Gamma$-module on the generators of $\pi_2(S_1)$ and $\pi_2(T_1)$. To describe $\pi_3(Y_n)$ requires more notation. Let $\sigma$ denote the generator of $\pi_2(S_1)$ and $\tau$ the generator of $\pi_2(T_1)$. For $x, y \in \pi_2(Y_n)$ let $S_{x,y}$ denote a 3-sphere, which is considered to be mapped to $\tilde{Y}$ by the Whitehead product $[x, y]$. Order the elements of $\Gamma$ arbitrarily as $\ldots, g^{-2}, g^{-1}, 1, g_1, g_2, \ldots$ subject to the condition that $g^{-n} = g^{-1}$.

Examining [19], we find

$$
\pi_3(Y_n) = \left( \bigoplus_{g \in \Gamma} \pi_3(S_g) \right) \oplus \left( \bigoplus_{g \in \Gamma} \pi_3(T_g) \right) \oplus \left( \bigoplus_{g \in \Gamma} \pi_3(S_{g\sigma, h\tau}) \right) \\
\oplus \left( \bigoplus_{g \in \Gamma} \pi_3(S_{g\sigma, h\alpha}) \right) \oplus \left( \bigoplus_{g \in \Gamma} \pi_3(S_{g\tau, h\tau}) \right).
$$

This is a free $\mathbb{Z}\Gamma$-module on the following set of generators:

1. A generator of $\pi_3(S_1)$
2. A generator of $\pi_3(T_1)$
3. $[\sigma, h\tau], h \in \Gamma$
4. $[\sigma, g\sigma], 1 < g$
5. $[\tau, g\tau], 1 < g$.

This follows since

$$
\bigoplus_{g, h \in \Gamma} \pi_3(S_{g\sigma, h\tau}) = \bigoplus_{h \in \Gamma} g \pi_3(S_{g\sigma, 1h\tau}) \\
= \bigoplus_{h \in \Gamma} g \pi_3(S_{g\sigma, h\tau}) \\
\cong \bigoplus_{h \in \Gamma} \mathbb{Z}\Gamma
$$

and

$$
\bigoplus_{g < h} \pi_3(S_{g\sigma, h\alpha}) = \bigoplus_{1 < a^{-1}b} \pi_3(S_{a\sigma, b\alpha}) \\
= \bigoplus_{1 < a^{-1}b} a \pi_3(S_{a, a^{-1}b\alpha}) \\
\cong \bigoplus_{1 < a^{-1}b} \mathbb{Z}\Gamma
$$

and similarly for $\bigoplus_{g < h} \pi_3(S_{g\tau, h\tau})$. □

A relative cell-complex structure on $(Y_n, \partial Y_n)$ is defined as follows (see figure 1). There are relative 1-cells $z_+$ in $S_+ \times I$, disjoint from $D_0$, and $z_-$ in

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$S_- \times I$, disjoint from the attaching discs of the 1-handles in $S_- \times \partial I$. Each disc $D_i$ is a relative 2-cell, so there are $2n + 1$ relative 2-cells. Finally, there is a relative 3-cell $\tau_+$ whose boundary is attached to $z_+$, $D_0$, and part of $\partial Y_n$, and there is a relative 3-cell $\tau_-$ whose boundary is attached to $z_-$, the union of the $D_i$'s, and part of $\partial Y_n$. For $i \geq 1$, $\tau_-$ meets both sides of $D_i$.

All homotopy groups of $Y_n$ are to be based in the simply connected subset $S_+ \times I$. Also, consider a map of a sphere into $Y_n$ which carries the basepoint into the simply-connected subset which is the union of $S_+ \times I$, $S_- \times I$, and the 1-handle attaching $C_0$ to $D_0$. By convention, we will use a path in this simply-connected subset to connect the image of the basepoint to $S_+ \times I$, and therefore such a map will represent a well-defined element of the homotopy group. With this convention in place, we will suppress basepoints from our notation.

Let $F = F_i$ be a circular homotopy of $Y_n$; that is, a homotopy from $Y_n \times I$ to $Y_n$ which starts and ends at the identity map. It will be assumed that all homotopies on $Y_n$ are the identity on the boundary at all times. Consider the difference obstructions between $F_i$ and the constant homotopy which is the identity map at each time $t$. To each of the two 1-cells of $Y_n$, $d_1(F)$ assigns an element of $\pi_2(Y_n)$, and to each 2-cell, $d_2(F)$ assigns an element of $\pi_3(Y_n)$. We have the following “Duality Theorem.”

**Theorem 3.2 (Duality Theorem)** Let $F$ and $G$ be circular homotopies of $Y_n$ which keep the boundary fixed at all times. If $d_1(F) = d_1(G)$, then $d_2(F) = d_2(G)$.

**PROOF.** For $1 \leq i \leq 2n$ let $\alpha_i$ be the free generator of $\pi_1(Y_n)$ which corresponds to the handle running from $C_i$ to $D_i$. It suffices to show that if $d_1(F) = 0$ then $d_2(F) = 0$, so assume that $F$ is constant on $z_+ \cup z_- \cup \partial Y_n$. Since $\tau_+$ deformation retracts into $z_+ \cup Y_n$, we may assume $F$ is constant on $\tau_+$ as well (so $d_2(F)(D_0) = 0$). Then $F_1$ is homotopic relative to $\tau_+ \cup z_- \cup \partial Y_n$ to modifications in $4n$ 3-balls, one on each side of each $D_i$ for $1 \leq i \leq 2n$. On one side (which side depends on orientation and sign conventions) the modification is by the element $d_3(F)(D_i)$ and on the other it is by $-d_2(F)(D_i)$. Since $F_1$ is the identity, its difference on $\tau_-$ is trivial. This difference, however, can be expressed (up to sign conventions) as

$$(\alpha_1 - 1)d_2(F)(D_1) + (\alpha_2 - 1)d_2(F)(D_2) + \cdots + (\alpha_{2n} - 1)d_2(F)(D_{2n}).$$

To see this, note that when the difference is calculated on $\tau_-$, it is necessary to move the modifications in balls which lie on one of the sides of the $D_i$'s across the 1-handles, and in the process of doing so, the element of $\pi_3(Y_n)$ used to construct the modification is acted upon by the generator of $\pi_1(Y_n)$.
determined by the 1-handle. From lemma 3.1, $\pi_3(Y_n)$ is a free $\mathbb{Z}\pi_1(Y_n)$-module, so we may regard this difference as an element in a direct sum of (infinitely many) copies of $\mathbb{Z}\pi_1(Y_n)$. In each copy, the coordinate of this element lies in the augmentation ideal. Since $\pi_1(Y_n)$ is free on the set $\{\alpha_1, \ldots, \alpha_{2n}\}$, its augmentation ideal is a free $\mathbb{Z}\pi_1(Y_n)$ module on $\{\alpha_1 - 1, \ldots, \alpha_{2n} - 1\}$ (see for example p. 70 of [30]). Therefore each $d_2(F)(D_i)$ must be trivial. □

4 Isotopies on $Y_k$

Here is a general construction for isotopies of 3-manifolds. Suppose that $M$ is a connected sum $P \# Q$ where the connected sum is taken along a 2-sphere $\Sigma$ containing the basepoint $x_0$. Replace $P$ with a 3-ball $D$, obtaining a manifold $M'$. We will build a 2-parameter family of diffeomorphisms of $M$, parameterized by the square $I^2 = I \times I$. Let $I^1 = \{(t,0) \mid 0 \leq t \leq 1\} \subseteq I^2$, and let $J^1 = \partial I^2 - I^1$. For each parameter in $J^1$, the diffeomorphism will be the identity. Now let $\tau$ be an element of $\pi_2(M', x_0)$. We may regard $\tau$ as a smooth family of imbeddings of $x_0$ into $M'$, parameterized by $I^2$. Since $M$ is parallelizable, this can be extended to a family of imbeddings of $D$, which are the inclusion for each parameter in $\partial I^2$. (This is also possible in the nonorientable case, for in the orientable double cover a family of imbeddings can be obtained which projects to a family in $M$.) The isotopy extension theorem allows us to extend these imbeddings of $D$ to a 2-parameter family of diffeomorphisms of $M'$. On $I^1$, this is a 1-parameter family of diffeomorphisms which is the identity on $D$. For this family, we may replace the copies of $D$ by copies of $P$ and extend using the identity on $P$ to obtain an isotopy $J_\ast$ of $M$.

It is instructive to fix a value of $t$ and examine what is happening for the parameters $\{(t, s) \mid 0 \leq s \leq 1\}$. At $(t, 1)$, the map of $M'$ is the identity. As $s$ decreases from 1 to 0, the ball $D$ moves around a loop $\beta_t$ in $M'$. Each loop $\beta_t$ is contractible, since as $t$ moves to 0 or 1, the $\beta_t$ deforms to $\beta_0$ or $\beta_1$ which are the constant loops at $x_0$. However, as $t$ moves from 0 to 1 the $\beta_t$ sweep out the element $\tau$ of $\pi_2(M', x_0)$. The map $J_\ast$ is constructed in three steps:

1. Replace $P$ by the ball $D$.
2. Choose an isotopy of $M'$ that slides $D$ around $\beta_t$.
3. Replace $D$ by $P$ at the final stage of this isotopy.

A diffeomorphism constructed in this way is called a slide diffeomorphism that slides $P$ around the loop $\beta_t$.

Any map of a 2-sphere into $Y_k$ which carries the basepoint into the union of $S_+ \times I$, $S_- \times I$, and the 1-handle from $C_0$ to $D_0$ determines an element of $\pi_2(Y_k)$. We denote the elements represented by the inclusions of $S_+ \times \{1/2\}$....
and $S_+ \times \{1/2\}$ by $\sigma_+$ and $\sigma_-$, respectively. Recall that $\pi_1(Y_k)$ is free on $2k$ generators. For $1 \leq i \leq 2k$, the 1-handle that goes from $C_i$ to $D_i$ determines a generator which will be denoted by $\alpha_i$.

Consider performing the construction described above to obtain an isotopy $F_t$ on $Y_k$. As usual, the boundary of $Y_k$ is to be held fixed by all isotopies and homotopies. It produces a 1-parameter family of slide diffeomorphisms, each of which slides $S_+ \times [0, 1/2]$ around a loop in $Y_k - S_+ \times [0, 1/2]$. These loops sweep out an immersion of a 2-sphere into $\pi_2(Y_k)$, and we choose this immersion as follows: map the boundary of a 2-disc to $z^+ \cap S_+ \times \{1/2\}$, map the fibers in a collar neighborhood of the boundary onto a path which travels through each 1-handle once, meeting in succession the 2-cells $C_0$, $D_0$, $C_1$, $D_1$, ..., $D_k$, and map the complement of the collar with degree 1 onto the 2-sphere $S_- \times 1/2$. This immersion represents the element $\alpha_1 \alpha_2 \cdots \alpha_{2k} \sigma_-$ of $\pi_2(Y_k)$.

The isotopy $F_t$ will be fundamental to our later constructions of isotopies in 3-manifolds. We must determine its difference elements on the 1-skeleton of $Y_k$. For $0 \leq j \leq 2k$, let $\beta_j = \alpha_1 \alpha_2 \cdots \alpha_j$; in particular, $\beta_0$ is the identity element of $\pi_1(Y_k)$. The obstructions $d_1(z^+)$ and $d_1(z^-)$ lie in $\pi_2(Y_k)$. By lemma 3.1, this is a free $\mathbb{Z}\pi_1(Y_k)$-module on the generators $\sigma_+$ and $\sigma_-$. 

**Proposition 4.1** There is an $\epsilon_1$ equal to 1 or $-1$, so that for the circular isotopy $F_t$ on $Y_k$,

$$d_1(z^+) = \beta_{2k} \sigma_- + \epsilon_1 \sum_{j=0}^{2k-1} (\beta_j \beta_{2k}^{-1} - \beta_{2k} \beta_j^{-1}) \sigma_+, \quad d_1(z^-) = -\beta_{2k}^{-1} \sigma_+ .$$

**Remark 4.2** Rather than trying to fix conventions and make a very delicate determination of $\epsilon_1$, we are allowing this ambiguity in the statement of proposition 4.1 and will structure our later arguments so that they work for either value of $\epsilon_1$.

**PROOF.** Regarding $F_t$ as a 1-parameter family of slide diffeomorphisms, we must determine the element of $\pi_1(Y_k)$ swept out by the images $F_t(z^+)$ and $F_t(z^-)$. To understand these elements, we use the following observation. Fix arcs $z'_+$ and $z'_-$ parallel to $z^+$ and $z^-$, respectively. Consider any map of a 2-disc into $Y_k$, which takes the boundary into the contractible subset which is the complement of $z^+_+ \cup z^-$ in the union of $S_+ \times I$, $S_- \times I$, and the 1-handle from $C_0$ to $D_0$. Put the map transverse to $z'_+ \cup z'_-$. For each point $p$ in the preimage of $z'_+$, choose any path from the boundary of the disc to $p$. Let $\alpha$ be the element
of $\pi_1(Y_k)$ represented by the image of this path. The point $p$ contributes $\pm \alpha \sigma_+$ to the element represented by the map. Similarly, each point in the preimage of $z_-$ contributes a corresponding multiple of $\sigma_-$. The element represented by the map is the sum of all of these.

In our situation, the discs will be $z_+ \times I$ and $z_- \times I$, and the maps will be the restrictions of $F_t$. Observe that the image of each $z_+ \times \{t\}$ is homotopic to the sliding path which determines the diffeomorphism $F_t$. We choose particularly simple sliding paths as illustrated in figure 2. These are imbedded except for exactly $4k$ values of $t$, where a portion of the sliding path sweeping around $S_- \times \{1/2\}$ cuts across a portion of an arc in $S_- \times I$ running from $D_j$ to $C_{j+1}$, for $0 \leq j \leq 2k - 1$.

Fix a value of $t$. When the filled in ball $D$ slides around this loop, during the construction of the $F_t$, the arc $z_+$ is dragged along and stretched out near the sliding path. When the sliding path has no self intersections, or near misses, $z_+$ simply ends up stretched close to the sliding path. For one value of $t$, near the time when the sliding path intersects $z'_-$, the image $F_t(z_+)$ will intersect $z'_-$. At this time, the image of a path in $z_+$ from either endpoint to the point whose image intersects $z'_-$ represents $\alpha_1 \cdots \alpha_{2k}$ and this makes a contribution of $\pm \beta_{2k} \sigma_-$ to $d_1(z_+)$. We choose orientation and sign conventions so that this element is $\beta_{2k} \sigma_-$. The remaining contributions arise due to self intersections of the sliding path, for then a portion of the stretched out sliding path will be pushed in front of $D$ when $D$ cuts back across the earlier part of the sliding path, and the image of $z_+$ under $F_t$ will loop back over $D$. This is illustrated in two-dimensional cross-section in figure 3. For one value of $t$ near the time when the sliding path has a self intersection, the image of $z_+$ will intersect $z'_+$ and make a contribution to the element $d_1(z_+)$.

In order to calculate these additional contributions, we consider two cases.
The first (illustrated in figure 3) occurs when the portion of the sliding path sweeping out \( \sigma_- \) cuts across the “earlier” portion of the sliding path that crosses from \( D_j \) to \( C_{j+1} \); the second case is when it cuts across the “later” portion when the sliding path is returning from \( C_{j+1} \) to \( D_j \). In the first case, \( D \) has slid through all \( 2k \) handles and into \( S_- \) to sweep out \( \sigma_- \), it starts to push a portion of the stretched out \( z_+ \) at a point where it has crossed the first \( i \) 1-handles; the portion before this point represents \( \alpha_1 \cdots \alpha_j = \beta_j \). This is pushed in front of \( D \) along a path representing \( \alpha_2^{-1} \cdots \alpha_1^{-1} = \beta_2^{-1} \), creating an intersection of \( F_t(z_+) \) with \( z'_+ \) so that a path in \( z_+ \) from an endpoint to the preimage of this intersection maps to \( \beta_j \beta_2^{-1} \). This contributes a term \( \epsilon_1 \beta_j \beta_2^{-1} \sigma_+ \) to \( d_1(z_+) \), where \( \epsilon_1 = \pm 1 \). For the second case, \( D \) slides over all \( 2k \) 1-handles, along a path representing \( \beta_{2k} \), then back over the last \( k-j \) 1-handles before encountering the image of \( z_+ \) at a point where it has crossed all \( 2k \) handles. The portion before this point represents \( \beta_{2k} \). The image of \( z_+ \) is then pushed backward through the first \( i \) handles, creating an intersection which contributes \( \pm \epsilon_1 \beta_{2k} \beta_j^{-1} \sigma_+ \) to \( d_1(z_+) \). Careful drawing of the pictures shows that this intersection occurs with the opposite intersection number from the one which created the term \( \epsilon_1 \beta_j \beta_2^{-1} \sigma_+ \), so the resulting formula for \( d_1(z_+) \) is the one given in proposition 4.1.

The behavior of \( z_- \) is much simpler, since it will only be moved by \( F_t \) whose sliding paths nearly intersect \( z_- \). There is only one such intersection, and at that time the sliding \( D \) meets \( z_- \) after it has passed over all \( 2k \) 1-handles, and pushes \( z_- \) backward over all \( 2k \) 1-handles. Careful drawing of the picture shows that this sign will be the opposite of the sign of the intersection of the image of \( z_+ \) with \( z_- \) that created the term \( \beta_{2k} \sigma_- \) in \( d_1(z_+) \), so \( d_1(z_-) = -\beta_{2k}^{-1} \sigma_+ \). □

Using theorem 3.2 and proposition 4.1, we can now calculate the difference elements on the 2-cells of \( Y_k \).

**Proposition 4.3** There is an \( \epsilon \) equal to 1 or \(-1\), so that for the circular isotopy \( F_t \) on \( Y_k \), and for each \( i \) with \( 0 \leq i \leq 2k \), the difference element of \( F_t \) on \( D_i \) is given by

\[
d_2(F_t)(D_i) = \beta_i^{-1}[\sigma_+, \beta_{2k} \sigma_-] + \epsilon \sum_{j=0}^{2k-1} \beta_i^{-1}[\sigma_+, \beta_{2k} \beta_j^{-1} \sigma_+] \\
- \epsilon \sum_{j=i}^{2k-1} \beta_i^{-1}[\sigma_+, \beta_j \beta_{2k}^{-1} \sigma_+].
\]

**Proof.** Let \( W \) denote a union of small disjoint collar neighborhoods of the \( D_i \) for \( 0 \leq i \leq 2k \). From theorem 3.2, \( d_2(F) \) equals \( d_2(G) \) for any circular
homotopy $G$ with $d_1(G) = d_1(F)$; we will construct $G$ in such a way that $d_2(G)$ can be easily determined. Consider a 1-parameter family of immersions of $z_+$ and $z_-$ realizing $d_1(F)$. Extend this using the inclusion maps on $\partial Y_k$ and $W$. By homotopy extension starting from the identity map we obtain a 1-parameter family $K_1$ of maps of $Y_k$. The map $K_1$ is the identity on $z_+ \cup z_- \cup W$. Applying lemma 2.2 and proposition 4.1 shows that the differences of $K_1$ on the 3-cells of $Y_k$ are given by:

\[
d_3(\tau_+) = [\sigma_+ , \beta_{2k} \sigma_- ] + \epsilon \left[ \sigma_+ , \sum_{j=0}^{2k-1} (\beta_j \beta_{2k}^{-1} - \beta_{2k} \beta_j^{-1}) \sigma_+ , \right],
\]
\[
d_3(\tau_-) = -[\sigma_- , \beta_{2k}^{-1} \sigma_+] .
\]

If we construct any homotopy $L$ from $K_1$ to the identity which keeps $z_+ \cup z_-$ fixed, then $KL$ will be a circular homotopy with $d_1(KL) = d_1(K) = d_1(F)$ and hence by theorem 3.2, $d_2(F) = d_2(KL) = d_2(L)$.

To construct $L$, we first deform $K_1$, keeping $\partial Y_k \cup W$ fixed, to a collection of modifications in balls (defined in section 2). On $\tau_+$ there is to be a modification on a ball for each of the following elements:

1. $[\sigma_+ , \epsilon \beta_j \beta_{2k}^{-1} \sigma_+ ]$, $0 \leq j \leq 2k - 1,$
2. $-[\sigma_+ , \epsilon \beta_{2k} \beta_j^{-1} \sigma_+ ]$, $0 \leq j \leq 2k - 1,$
3. $[\sigma_+ , \beta_{2k} \sigma_- ]$.

On $\tau_-$, there is a modification in a ball by $-[\sigma_- , \beta_{2k}^{-1} \sigma_+]$.

Slide each modification of type (2) around the loop $\beta_{2k} \beta_j^{-1}$, changing it to a modification in a ball in $\tau_+$ by the element $-\epsilon \beta_j \beta_{2k} \sigma_+$. Since this equals $-[\sigma_+ , \epsilon \beta_j \beta_{2k}^{-1} \sigma_+]$, it annihilates a corresponding term of type (1). During these homotopies, the $D_i$ move through difference elements of the form $[\beta^{-1} \sigma_+ , \pm \epsilon \beta^{-1} \beta_{2k} \beta_i \sigma_+]$, where $\beta$ is the element represented by the initial segment of the sliding path from its starting point until the encounter with $D_i$ that contributes that element to the difference. The sign before the $\epsilon$ depends on the choices and definitions that have been made to define difference obstructions, but the sign will differ according to whether the sliding direction agrees or disagrees with the positive normal direction to $D_i$ (which is the direction pointing into $S_+ \times I$). Put $\epsilon$ equal to the product of $\epsilon_1$ and the sign obtained when the sliding direction agrees with the positive normal direction to $D_i$.

The oriented sliding loop $\beta_{2k} \beta_j^{-1}$ for the modification $-[\sigma_+ , \epsilon \beta_{2k} \beta_j^{-1} \sigma_+]$ intersects each $D_i$ once with positive orientation, with initial segment $\beta_i$. For each $i$ with $i \leq j$, it intersects a second time with negative orientation, with initial segment $\beta_{2k} \beta_j^{-1} \beta_i$. Thus for every $i$ sliding that modification will contribute
\(\varepsilon[\beta_i^{-1}\sigma_+, \beta_i^{-1}\beta_{2k}\beta_j^{-1}\sigma_+]\) to the difference on \(D_i\), while for \(i \leq j\) there will be a second contribution of \(-\varepsilon[\beta_i^{-1}\beta_{2k}\sigma_+, \beta_i^{-1}\sigma_+]\). Summing these gives all except the initial term on the right-hand side of the formula in the proposition.

To complete \(L\), the type (3) modification must be slid around the loop \(\beta_{2k}\) to become a modification on \(\tau_-\). When it passes through each \(D_i\), that \(D_i\) picks up the difference \([\beta_i^{-1}\sigma_+, \beta_i^{-1}\beta_{2k}\sigma_-]\). After this sliding is completed, the resulting modification by \([\beta_{2k}^{-1}\sigma_+, \sigma_-]\) cancels with the modification by \(-[\sigma_-, \beta_{2k}^{-1}\sigma_+]\) on \(\tau_-\). \(\square\)

5 The main diagram

Let \(P_1, \ldots, P_t\) be the nonsimply connected irreducible prime summands of \(M\), let \(s\) be the number of \(S^2 \times S^1\)-summands, and let \(Q_1, \ldots, Q_t\) be the simply connected prime summands. Since we are assuming that \(M\) has no 2-sphere boundary components, the simply connected prime summands are homotopy 3-spheres (i.e., not \(D^3\)). We regard \(M\) as constructed in the following way. From \(S^3\), obtain a punctured 3-cell \(B\) by removing the interiors of disjoint 3-balls \(B_i^+, B_i^-, B_2^+, \ldots, B_r^+, E_1, \ldots, E_r, E_{r+1}, \ldots, E_{r+t}\) from \(S^3\).

Attach an \(S^2 \times [0, 1]\) to the boundaries of \(B_i^+\) and \(B_i^-\) to produce an \(S^2 \times S^1\) summand for each \(1 \leq i \leq s\), and replace the interior of each \(E_i\) by \(P_i - D^3\), when \(i = r\), or by \(Q_j - D^3\), when \(i = r + j\). We use a point \(x_0\) in \(B\) as the basepoint of \(M\). Any path with endpoints in \(B\) determines a unique element of \(\pi_1(M, x_0)\).

Let \(\Sigma\) be a 1-point union \(\Sigma_1 \vee \Sigma_2\) of two 2-spheres. For the different cases of the proof, we will use various imbeddings of \(\Sigma\) into \(B\), such that the join point equals \(x_0\). There is a restriction fibration \(\text{Maps}(\Sigma_1 \vee \Sigma_2, M \rel x_0) \to \text{Maps}(\Sigma_1 \vee \Sigma_2, M) \to M\) where the projection is evaluation at \(x_0\), and for \(A \subset M\), \(\text{Maps}(A, M)\) denotes the connected component of the inclusion in the space of maps from \(A\) to \(M\). Note that \(\pi_q(\text{Maps}(\Sigma_1 \vee \Sigma_2, M \rel x_0)) \cong \pi_{q+2}(M, x_0) \oplus \pi_{q+2}(M, x_0)\) where the \(i^{th}\) coordinate is given by the difference element on \(\Sigma_i\) for \(i = 1, 2\). The restriction fibration gives the exact sequence in the middle row of the following main diagram,

\[
\begin{array}{c}
\pi_2(M, x_0) \longrightarrow \pi_1(\text{Diff}(M \rel x_0)) \longrightarrow \pi_1(\text{Diff}(M)) \\
\downarrow = \quad \downarrow \rho \quad \quad \downarrow \\
\pi_2(M, x_0) \longrightarrow \pi_1(\text{Maps}(\Sigma_1 \vee \Sigma_2, M \rel x_0)) \longrightarrow \pi_1(\text{Maps}(\Sigma_1 \vee \Sigma_2, M)) \\
\downarrow = \quad \downarrow d_2 \quad \quad \downarrow d_2 \\
\pi_2(M, x_0) \longrightarrow \pi_3(M, x_0) \oplus \pi_3(M, x_0) \longrightarrow \pi_3(M, x_0) \oplus \pi_3(M, x_0) \\
\end{array}
\]

\[
\partial(\pi_2(M, x_0))
\]
where $\rho$ is induced by restriction and $d_2$ denotes the difference on $\Sigma_1 \vee \Sigma_2$. By lemma 2.1, $\partial(\eta) = ([\langle \Sigma_1 \rangle, \eta], [\langle \Sigma_2 \rangle, \eta])$, where $\langle \Sigma_i \rangle$ is a generator of $\pi_2(\Sigma_i)$ for $i=1,2$.

In the succeeding sections, we will construct elements $f_k$ in $\pi_1(\text{Diff}(M,x_0))$, $k \geq 1$. Each $f_k$ will look like one of the isotopies studied in section 4, supported in an imbedded copy of some $Y_m$ in $M$, where $m$ will be either 0 or $k$. The propositions in section 4 will allow us to read off the difference elements $d_2\rho(f_k)$ in $\pi_3(M,x_0) \oplus \pi_3(M,x_0)$. The main work in the proof is to show that the difference elements $d_2\rho(f_k)$ are linearly independent modulo the image of $\partial$. This shows that the images of the $f_k$ in the abelian group $\pi_1(\text{Diff}(M))$ are linearly independent, and hence that $\pi_1(\text{Diff}(M))$ is infinitely generated.

To work in $\pi_3(M)$ we use a description of it from V.1.2 of Hendriks’ dissertation [15]. It involves the $\mathbb{Z}$-symmetric product of $\pi_2(M)$, where the $R$-symmetric product of an $R$-module $T$ is defined to be the $R$-module $S(T) = (T \otimes_R T)/( t_1 \otimes t_2 \sim (t_2 \otimes t_1)$. Note that any symmetric $R$-bilinear homomorphism from $T \times T$ to an $R$-module $A$ induces an $R$-module homomorphism from $S(T)$ to $A$. In our constructions, the induced homomorphism will not generally be $\mathbb{Z}\pi_1(M)$-linear, because the natural $\mathbb{Z}\pi_1(M)$-action on the symmetric product $S(\pi_2(M))$ is given by $\gamma(\sigma \otimes \tau) = \gamma\sigma \otimes \gamma\tau$.

**Theorem (Hendriks)** There is an exact sequence of $\pi_1(M,x_0)$-module homomorphisms

$$0 \to S(\pi_2(M,x_0)) \xrightarrow{Wh} \pi_3(M,x_0) \xrightarrow{v} \pi_2(M,x_0)/2\pi_2(M,x_0) \to 1$$

where $Wh(x \otimes y) = [x,y]$, $\text{Hopf}(x)$ means the Hopf map from $S^3$ to $S^2$ followed by $x$, and $v(\text{Hopf}(x)) = x + 2\pi_2(M,x_0)$.

In later sections, we will usually denote the elements of $S(T)$ by $[\sigma,\tau]$, in analogy with the (even-dimensional) Whitehead product.

6  The case of $s \geq 2$

Throughout this section assume that $s \geq 2$. The first $s$ summands of $M$ are $S^2 \times S^1$, with fundamental groups generated by $\gamma_i$ for $1 \leq i \leq s$, the next $r$ summands $P_{s+1}, P_{s+2}, \ldots, P_{s+r}$ ($r \geq 0$) are irreducible and not simply connected, and the remaining $t$ summands are homotopy 3-spheres. For $s+1 \leq$
Fig. 4. The imbedding of $Y_0$ in $M$

$i \leq s + r$, define an element $\Sigma(i) \in \mathbb{Z}\pi_1(M)$ by

$$\Sigma(i) = \begin{cases} 0 & \text{if } \pi_1(P_i) \text{ is infinite} \\ \sum_{g \in \pi_1(P_i)} g & \text{if } \pi_1(P_i) \text{ is finite} \end{cases}$$

There is a $\mathbb{Z}\pi_1(M)$-module presentation for $\pi_2(M)$ with generators $\sigma_1, \ldots, \sigma_{r+s}$ and relations

(R1) $\Sigma(i)\sigma_i = 0$ for $s + 1 \leq i \leq s + r$, and
(R2) $\sum_{i=1}^{s}(1 - \gamma_i)\sigma_i + \sum_{i=s+1}^{s+r} \sigma_i = 0$.

This presentation is given in [45] in the closed case. Another derivation, which applies to any compact 3-manifold, is given less explicitly on p. 110 of [15].

Fix a positive integer $k$, and imbed $Y_0$ in $M$ as indicated in figure 4. That is, the image is disjoint from the basepoint $x_0$, and satisfies the following.

1. $S_+ \times I$ is a collar neighborhood of $\partial B_1^+$ in $B$.
2. $S_- \times I$ is a collar neighborhood of $\partial B_1^-$ in $B$.
3. The 1-handle from $C_0$ to $D_0$ determines the element $\gamma_2^k$ in $\pi_1(M)$.

Under the induced homomorphism from $\pi_2(Y_0)$ to $\pi_2(M)$, $\sigma_+$ maps to $\sigma_1$ and $\sigma_-$ maps to $-\gamma_2^k\gamma_1\sigma_1$. The last observation uses our convention that the inclusion of $S_2^- \times \{\frac{1}{2}\}$ which represents $\sigma_-$ is regarded as an element of $\pi_2(Y_0)$ by using a path in the 1-handle from $C_0$ to $D_0$ to make it into a based map into $S_2^+ \times I$; this 1-handle represents $\gamma_2^k$ in $\pi_1(M)$.

Imbed $\Sigma$ in $B$ so that

4. $\Sigma_1$ is isotopic to $\partial B_1^-$ and $\Sigma_1 \cap Y_0 = -D_0$.
5. $\Sigma_2$ is isotopic to $\partial B_1^+$ and $\Sigma_2 \cap Y_0 = C_0$.

The notation $-D_0$ indicates that the chosen normal orientation on $\Sigma_1$ disagrees with the positive normal orientation on $D_0$, which points into $S_2^+ \times I$. In $\pi_2(M)$, $\langle \Sigma_1 \rangle$ represents $\sigma_1$ and $\langle \Sigma_2 \rangle$ represents $-\gamma_1\sigma_1$.

Let $f_k$ be the isotopy on $Y_0$ constructed in section 4, regarded as a loop in
$\pi_1(\text{Diff}(M, x_0))$. The element $d_2\rho(f_k)$ can be now read off using proposition 4.3:

\[
d_2\rho(f_k)(\Sigma_1) = -d_2(D_0) = [\sigma_1, \gamma_2^k \gamma_1 \sigma_1]
d_2\rho(f_k)(\Sigma_2) = d_2(C_0) = \gamma_2^{-k} d_2(D_0) = [\gamma_2^{-k} \sigma_1, -\gamma_1 \sigma_1]
\]

Suppose that some linear combination $\sum_{k=1}^n m_k f_k$ is trivial in $\pi_1(\text{Diff}(M))$. By the main diagram in section 5, there exists $\eta \in \pi_2(M)$ so that

\[
[-\gamma_1 \sigma_1, \eta] = [\sigma_1, \sum_{k=1}^n m_k \gamma_2^k \gamma_1 \sigma_1]
\]

\[
[\sigma_1, \eta] = \left[ \sum_{k=1}^n m_k \gamma_2^{-k} \sigma_1, -\gamma_1 \sigma_1 \right]
\]

All of the elements appearing in these equations actually lie in the subgroup $S(\pi_2(M))$. Define a ring homomorphism (not a $\mathbb{Z}\pi_1(M)$-module homomorphism) from $S(\pi_2(M))$ to $\mathbb{R}$ by sending $\sigma_1$ to 1, $\sigma_2$ to $-1$, and all other $\sigma_i$ to 0, and sending each of $\gamma_1$ and $\gamma_2$ to a transcendental number $\tau$ and all other generators of $\pi_1(M)$ to 1. Applying it to the above equations, and denoting the image of $\eta$ by $\bar{\eta}$, we have

\[
-\tau \bar{\eta} = \sum_{k=1}^n m_k \tau^{k+1}
\]

\[
\bar{\eta} = -\sum_{k=1}^n m_k \tau^{1-k}
\]

in $\mathbb{R}$. Multiplying the first equation by $\tau^{-1}$ and adding, we obtain

\[
0 = \sum_{k=1}^n m_k (\tau^k - \tau^{1-k})
\]

Since $\tau$ is transcendental, all $m_k$ must equal 0. This completes the proof of part (b) of the Main Theorem in the case $s \geq 2$.  

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7 The case of $s=0$

Throughout this section, assume that $s=0$ and $r \geq 3$. The first $r$ summands of $M$ are $P_1, \ldots, P_r$ which are irreducible and not simply connected, and the last $t$ are homotopy 3-spheres. A $\mathbb{Z}\pi_1(M)$-module presentation for $\pi_2(M)$ has generators $\sigma_1, \ldots, \sigma_r$ and relations

(R1) $\Sigma(i)\sigma_i=0$ for $1 \leq i \leq r$, and
(R2) $\Sigma_{i=1}^r \sigma_i=0$.

It will be convenient to simplify the presentation of $\pi_2(M)$ by eliminating $\sigma_r$ so that the relations become

(R1) $\Sigma(i)\sigma_i$, $1 \leq i \leq r-1$, and
(R2) $\Sigma(r)(\Sigma_{i=1}^{r-1} \sigma_i)=0$.

Choose nonzero elements $\mu_2$ and $\mu_3$ in $\pi_1(P_2)$ and $\pi_1(P_3)$. For each $k$, there is an imbedded $Y_k$ in $M$, as illustrated in figure 5, with the following properties:

(1) $S_+ \times I$ is a neighborhood of $\partial D_1$.
(2) $S_- \times I$ is a neighborhood of $\partial D_2$.
(3) The 1-handle from $C_0$ to $D_0$ lies in $B$.
(4) Each 1-handle from $C_{2j-1}$ to $D_{2j-1}$ represents $\mu_2$, for $1 \leq j \leq k$.
(5) Each 1-handle from $C_{2j}$ to $D_{2j}$ represents $\mu_3$, for $1 \leq j \leq k$.

Under the imbedding of $Y_k$ into $M$, the generators $\sigma_+$ and $\sigma_-$ map to $\sigma_1$ and $\sigma_2$ respectively. Each $\alpha_{2j-1}$ in $\pi_1(Y_k)$ maps to $\mu_2$, and each $\alpha_{2j}$ maps to $\mu_3$. Thus $\beta_{2j}$ maps to $(\mu_2\mu_3)^j$ and $\beta_{2j+1}$ maps to $(\mu_2\mu_3)^j\mu_2$. In slight abuse of notation, we use $\beta_{2j}$ and $\beta_{2j+1}$ to denote these images.

Imbed $\Sigma$ in $B$ so that

(6) $\Sigma_1$ is isotopic to $\partial E_1$ and $\Sigma_1 \cap Y_k=C_0$.
(7) $\Sigma_2$ is isotopic to $\partial E_2$ and $\Sigma_2 \cap Y_k=-D_0 \cup C_2 \cup -D_2 \cup C_4 \cup \cdots \cup C_{2k} \cup -D_{2k}$.

In item (7), the notation $-D_i$ indicates that the chosen (outward normal) orientation on $\Sigma_2$ disagrees with the positive normal of $D_i$, which points into
\( S \times I \). In \( \pi_2(M) \), \( \langle \Sigma_1 \rangle \) represents \( \sigma_1 \) and \( \langle \Sigma_2 \rangle \) represents \( \sigma_2 \).

Consider the isotopy on \( Y_k \) constructed in section 4. Since \( C_j \) and \( D_j \) are parallel, the difference on \( C_j \) equals the result of acting on the difference on \( D_k \) by the generator \( \alpha_k \) (where \( \alpha_0 \) is the identity element). Using proposition 4.3 we calculate

\[
d_2(\Sigma_1) = d_2(C_0) \\
= d_2(D_0) \\
= [\sigma_1, \beta_2 k \sigma_2] + [*\sigma_1, *\sigma_1]
\]

\[
d_2(\Sigma_2) = -d_2(D_0) + \sum_{j=1}^{k} (d_2(C_{2j}) - d_2(D_{2j})) \\
= -d_2(D_0) + \sum_{j=1}^{k} (\mu_3 - 1)d_2(D_{2j}) \\
= -[\sigma_1, \beta_2 k \sigma_2] + \sum_{j=1}^{k} (\mu_3 - 1)\beta_2^{-1}[\sigma_1, \beta_2 k \sigma_2] + [*\sigma_1, *\sigma_1]
\]

\[
= \sum_{j=0}^{2k} (-1)^{j+1} \beta_2^{-1}[\sigma_1, \beta_2 k \sigma_2] + [*\sigma_1, *\sigma_1]
\]

where \([*\sigma_1, *\sigma_1]\) denotes some sum of elements each of the form \([\delta_1 \sigma_1, \delta_2 \sigma_1]\).

Let \( A \) be the quotient of \( \mathbb{Z} \pi_1(M) \) by the ideal generated by \( \{ \Sigma(1), \ldots, \Sigma(r) \} \).

For \( 1 \leq i, j \leq r - 1 \), define \( \psi_{i,j} : \pi_2(M) \times \pi_2(M) \to A \) by

\[
\psi_{i,j}(a_1 \sigma_1 + \cdots + a_{r-1} \sigma_{r-1}, b_1 \sigma_1 + \cdots + b_{r-1} \sigma_{r-1}) = a_i b_j + b_i a_j
\]

This is \( \mathbb{Z} \)-bilinear and symmetric, so defines a \( \mathbb{Z} \)-linear homomorphism from \( S(\pi_2(M)) \) to \( A \).

For each \( k \geq 1 \), let \( f_k \) be the element of \( \pi_1(\text{Diff}(M, x_0)) \) represented by the isotopy supported on \( Y_k \). Suppose for some \( n \) and some integers \( m_k \) that \( \sum_{k=1}^{n} m_k f_k \) is trivial as an element of \( \pi_1(\text{Diff}(M)) \). From the main diagram, there must exist an element \( \eta \in \pi_2(M) \) so that

\[
[\sigma_1, \eta] = d_2 \left( \sum_{k=1}^{n} m_k f_k \right)(\Sigma_1)
\]

\[
[\sigma_2, \eta] = d_2 \left( \sum_{k=1}^{n} m_k f_k \right)(\Sigma_2)
\]

and therefore

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\[ [\sigma_1, \eta] = [\ast \sigma_1, \ast \sigma_1] + \sum_{k=1}^{n} m_k[\sigma_1, \beta_{2k}\sigma_2] \]

\[ [\sigma_2, \eta] = [\ast \sigma_1, \ast \sigma_1] + \sum_{k=1}^{n} m_k \sum_{j=0}^{2k} (-1)^{j+1}[\beta_j^{-1}\sigma_1, \beta_j^{-1}\beta_{2k}\sigma_2] \]

in \( \pi_3(M) \). All elements appearing in these equations actually lie in the subgroup \( S(\pi_2(M)) \). Writing \( \eta = \sum_{i=1}^{r-1} c_i \sigma_i \) and applying \( \psi_{1,2} \) to the first equation and \( \psi_{2,2} \) to the second yields

\[
c_2 = \sum_{k=1}^{n} m_k \beta_{2k} \\
2c_2 = 0
\]

as elements of \( A \), and therefore \( \sum_{k=1}^{n} 2m_k\beta_{2k} = 0 \) in \( A \). In summary, linear independence of the \( f_k \) is implied if \( \sum_{k=1}^{n} 2m_k(\mu_2\mu_3)^k = 0 \) in \( A \) only when all \( m_k = 0 \).

To verify this for most cases, we will find a representation of \( A \) into the endomorphism ring of a vector space, so that the image of \( \sum_{k=1}^{n} 2m_k\beta_{2k} \) is zero only when all \( m_k = 0 \). For summands that have finite fundamental group, we will use the hypothesis that their fundamental groups are isomorphic to subgroups of \( SO(4) \) which act freely on \( S^3 \).

Write \( \pi_1(M) = G_1 \ast \cdots \ast G_q \ast F_1 \ast F_2 \ast \cdots \ast F_{r-q} \), where each \( G_i \) and \( F_j \) is the fundamental group of a nonsimply connected prime summand of \( M \), each \( G_i \) is infinite, and each \( F_i \) is finite. Put \( G = G_1 \ast \cdots \ast G_q \) and \( F = F_1 \ast \cdots \ast F_{r-q} \). Let \( V \) be the vector space \( V = \oplus_{\sigma \in G} \mathbb{R}^4 \), and denote a point in \( V \) by \( (v_{g_1}, v_{g_2}, \ldots) \) where \( g_1 = 1, g_2, g_3, \ldots \), is an enumeration of \( G \) and each \( v_{g_i} \in \mathbb{R}^4 \). In case \( q = 0 \), \( V = \mathbb{R}^4 \). For any choice of representations of the \( F_i \) into \( SO(4) \) (and these choices will later be varied according to cases we shall consider), there is a \( \pi_1(M) \)-action on \( V \) defined on generators as follows. If \( g \in G_i \) and \( f \in F_j \) define \( gf(v_{g_1}, v_{g_2}, \ldots) \) to be the element of \( V \) whose \( gg_k \)-coordinate is \( f(v_{g_k}) \).

For all \( v \in V \) and \( h \in F_i \), we have

\[
h \left( \sum_{f \in \Sigma(i)} f(v) \right) = \sum_{f \in \Sigma(i)} hf(v) = \sum_{f \in \Sigma(i)} f(v) = \left( \sum_{f \in \Sigma(i)} f \right)(v).
\]

Assuming that the representations of the \( F_i \) are chosen so that each \( F_i \) acts freely on \( S^3 \), this implies that \( \Sigma(i) \) is the zero endomorphism of each \( \mathbb{R}^4 \) factor of \( V \) and hence of \( V \) (actually, all that would be necessary would be that the elements have no common fixed point). Therefore there is a \( \pi_1(M) \)-
module homomorphism $\Psi$ from $A$ to $\text{End}(V)$. Let $v_1$ denote the element \(((1, 0, 0, 0), (0, 0, 0, 0), (0, 0, 0, 0), \ldots)\) of $V$.

Suppose that at least one irreducible summand has infinite fundamental group. We may assume that this is $\pi_1(P_2)$. Since the fundamental group of a compact orientable irreducible 3-manifold must be either finite or torsionfree, $\mu_2$ must have infinite order. Then, the action of $\sum_{k=1}^n 2m_k (\mu_2 \mu_3)^k$ on $V$ sends $v_1$ either to an element with $m_k \mu_k^3 (1, 0, 0, 0)$ in the coordinate corresponding to $\mu_2^k$ (if $\pi_1(P_3)$ is finite), or to an element with $(m_k, 0, 0, 0)$ in the coordinate corresponding to $(\mu_2 \mu_3)^k$ (if $\pi_1(P_3)$ is infinite), hence $\Psi(\sum_{k=1}^n 2m_k (\mu_2 \mu_3)^k)$ cannot act trivially on $V$ unless all $m_k = 0$. This completes the proof in the case when some $\pi_1(P_i)$ is infinite, so from now on, we assume that all $\pi_1(P_i)$ are finite.

Suppose first that there are at least two irreducible summands which have fundamental group of order greater than two, and therefore we may choose $\mu_2$ and $\mu_3$ to be elements which are not $-I$ (under chosen representations to $\text{SO}(4)$ whose images act freely on $S^3$). Using lemma 2.3, we may choose the representation of $\pi_2(P)$ into $\text{SO}(4)$ so that for some coordinates on $\mathbb{R}^4$ as $\mathbb{C} \times \mathbb{C}$, the element $\Psi(\mu_2 \mu_3)$ has the form

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$$

and $e^{i\theta}$ or $e^{i\varphi}$ is a transcendental complex number. Thus $\Psi\left(\sum_{k=1}^n 2m_k (\mu_2 \mu_3)^k\right)$ is

$$\begin{pmatrix} \sum_{k=1}^n 2m_k (e^{i\theta})^k & 0 \\ 0 & \sum_{k=1}^n 2m_k (e^{i\varphi})^k \end{pmatrix}$$

and is zero only when all $m_k = 0$.

The remaining case is that all summands have finite fundamental group, and at most one of them has fundamental group not of order two. We assume that $\pi_1(P_i)$ is a finite group $F$, possibly of order two, and that $\pi_1(P_i)$ is of order two for $i \geq 2$. We will construct a $\pi_1(M)$-module homomorphism from $\pi_2(M)$ to a certain $\pi_1(M)$-module structure on $\mathbb{R}^6$. To set notation, write $\pi_1(M) = F \ast (\mathbb{Z}/2)^{r-1}$ where the $i^{th}$ $\mathbb{Z}/2$ factor is generated by an involution $\tau_i$ and $r \geq 3$. There is a $\mathbb{Z}\pi_1(M)$-module presentation

$$\pi_2(M) = \mathbb{Z}\pi[\sigma_1, \ldots, \sigma_r] / \langle \Sigma(1) \sigma_1, (1 + \tau_i) \sigma_i \rangle : (2 \leq i \leq r), \Sigma_{i=1}^r \sigma_i \rangle .$$
Define a $\pi_1(M)$-module structure on $\mathbb{R}^6$ as follows. Choose a representation $\phi: F \to \text{SO}(4)$ so that $\Sigma(1)$ acts as the zero endomorphism, and make the following assignments (where $I_\ell$ denotes the $\ell \times \ell$ identity matrix and $\text{diag}(A_1, \ldots, A_n)$ denote a block diagonal matrix).

\[
\tau_2 \mapsto \begin{pmatrix} I_3 \\ -\cos(\theta) - \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \\ -1 \end{pmatrix}, \quad \tau_3 \mapsto \text{diag}(I_3, -1, 1, -1),
\]

$\tau_i \mapsto I_6$ for $i \geq 4$, and $f \mapsto \begin{pmatrix} \phi(f) & 0 \\ 0 & I_2 \end{pmatrix}$ for $f \in F$. Note that

\[
(\tau_2 \tau_3)^k \mapsto \begin{pmatrix} I_3 & 0 \\ \cos(k\theta) & -\sin(k\theta) \\ \sin(k\theta) & \cos(k\theta) \\ 1 \end{pmatrix},
\]

and $(\tau_3 - 1) \mapsto \text{diag}(0_3, -2, 0, -2)$ where $0_3$ is the $3 \times 3$ zero matrix. Let $\mu$ denote $\tau_2 \tau_3$. Denoting the standard basis of $\mathbb{R}^6$ by $\{e_1, \ldots, e_6\}$, we have $\mu^\pm(e_4) = \cos(k\theta)(e_4) \pm \sin(k\theta)e_5$, $(\mu^k - \mu^{-k})(e_4) = 2 \sin(k\theta)e_5$, and for any $v \in \mathbb{R}^6$, $(\tau_3 - 1)(v)$ is of the form $a_4e_4 + a_6e_6$.

A $\pi_1(M)$-homomorphism $\Psi$ from $\pi_2(M)$ to $\mathbb{R}^6$ is defined by sending $\sigma_1$ to $e_4$, $\sigma_2$ to $e_6$, $\sigma_3$ to $-e_4 - e_6$, and $\sigma_i$ to 0 for $i \geq 4$. This determines a $\mathbb{Z}$-homomorphism $S(\Psi): S(\pi_2(M)) \to S(\mathbb{R}^6)$, where as usual $S(X)$ denotes the symmetric product of the module $X$.

Define $\mathbb{R}$-homomorphisms $\psi_{i,j}: S(\mathbb{R}^6) \to \mathbb{R}$ by $\psi_{i,j}([\sum a_i e_k, \sum b_j e_j]) = a_i b_j + b_i a_j$. For any $\omega_1$ and $\omega_2$ in $\pi_2(M)$, we have

\[
\psi_{4,5}(\Psi)([(\tau_3 - 1)\omega_1, (\tau_3 - 1)\omega_2]) = \psi_{4,5}([a_4 e_4 + a_6 e_6, b_4 e_4 + b_6 e_6]) = 0.
\]

For this case we will need more precise information about the difference obstruction. From proposition 4.3 we have
\[ d_2(f_k)(\Sigma_2) = -d_2(f_k)(D_0) + \sum_{j=1}^{k} (d_2(f_k)(C_{2j}) - d_2(f_k)(D_{2j})) \]
\[ = -[\sigma_1, \mu^k \sigma_2] - \epsilon \sum_{j=1}^{k} [\sigma_1, (\mu^j - \mu^{-j}) \sigma_1] \]
\[ + \epsilon \sum_{j=1}^{k} ((\tau_3 - 1) d_2(f_k)(D_{2j})) \]

and hence

\[ d_2 \left( \sum_{k=1}^{n} m_k f_k \right)(\Sigma_2) = - \sum_{k=1}^{n} m_k \left( -[\sigma_1, \mu^k \sigma_2] - \epsilon \sum_{j=1}^{k} [\sigma_1, (\mu^j - \mu^{-j}) \sigma_1] \right) \]
\[ + \epsilon \sum_{k=1}^{n} \sum_{j=1}^{k} m_k ((\tau_3 - 1)d_2(f_k)(D_{2j})) \]

If \( \sum_{k=1}^{n} m_k f_k \) is trivial in \( \pi_1(\text{Diff}(M)) \), then from the main diagram in section 5 this difference must equal \([\sigma_2, \eta]\) for some \( \eta \in \pi_2(M) \). Write \( \Psi(\eta) = \sum \eta_i e_i \). Applying \( S(\Psi) \) to \([\sigma_2, \eta]\) and the expression above for \( d_2(\sum_{k=1}^{n} m_k f_k)(\Sigma_2) \) yields the equation

\[ [e_6, \sum \eta_i e_i] = \sum_{k=1}^{n} m_k \left( -[e_4, e_6] - \epsilon \sum_{j=1}^{k} [e_4, 2 \sin(j\theta) e_5] \right) \]
\[ + \sum_{k=1}^{n} \sum_{j=1}^{k} m_k \Psi((\tau_3 - 1)d_2(f_k)(D_{2j})) \]

in \( S(\mathbb{R}^6) \). Applying \( \psi_{4,5} \), which as noted above annihilates all terms of the form \( S(\Psi)((\tau_3 - 1)[\omega_1, \omega_2]) \), we have

\[ 0 = -2\epsilon \sum_{k=1}^{n} \sum_{j=1}^{k} m_k \sin(j\theta) = -2\epsilon \sum_{j=1}^{n} \left( \sum_{k=j}^{n} m_k \right) \sin(j\theta) \]

Since \( \theta \) was arbitrary and the functions \( \sin(j\theta) \) are linearly independent over \( \mathbb{R} \), this shows that all \( m_k = 0 \), completing the proof of part (b) of the Main Theorem in the case \( s = 0 \).
Throughout this section assume that $s = 1$ and $r \geq 1$. The first summand of $M$ is $S^2 \times S^1$, with fundamental group generated by $\gamma$, the next $r$ summands $P_1, \ldots, P_r$ are irreducible and not simply connected, and the other prime summands are homotopy 3-spheres. There is a $\mathbb{Z}\pi_1(M)$-module presentation for $\pi_2(M)$ with generators $\sigma_0, \ldots, \sigma_r$ and relations

(R1) $\Sigma(i)\sigma_i = 0$ for $1 \leq i \leq r$, and
(R2) $(1 - \gamma)\sigma_0 + \sum_{i=1}^r \sigma_i = 0$.

Imbed $Y_0$ (which is simply two $S^2 \times I$'s connected by a 1-handle) in $M$, disjoint from the basepoint $x_0$, so that

1. $S_+ \times I$ is a collar neighborhood of $\partial B_1^+$ in $B$.
2. $S_- \times I$ is a collar neighborhood of $\partial B_1^-$ in $B$.
3. The 1-handle from $C_0$ to $D_0$ determines an element $\alpha \in \pi_1(P_1 \ast \cdots \ast P_r)$.

Under the induced homomorphism from $\pi_2(Y_0)$ to $\pi_2(M)$, $\sigma_+$ maps to $\sigma_0$ and $\sigma_-$ maps to $-\gamma \sigma_0$.

Imbed $\Sigma$ in $B$ so that

4. $\Sigma_1$ is isotopic to $\partial B_1^-$ and $\Sigma_1 \cap Y_0 = -D_0$.
5. $\Sigma_2$ is isotopic to $\partial B_1^+$ and $\Sigma_2 \cap Y_0 = C_0$.

As usual, the notation $-D_0$ indicates that the normal orientation on $\Sigma_1$ disagrees with the positive normal orientation on $D_0$, which points into $S^2 \times I$. In $\pi_2(M)$, $\langle \Sigma_1 \rangle$ represents $\sigma_1$ and $\langle \Sigma_2 \rangle$ represents $-\gamma \sigma_1$.

Let $f_\alpha$ be the isotopy on $Y_0$ constructed in section 4, regarded as a loop in $\pi_1(Diff(M, x_0))$. The element $d_2\rho(f_\alpha)$ can be now read off using proposition 4.3:

\[
d_2\rho(f_\alpha)(\Sigma_1) = -d_2(D_0) = [\sigma_0, \alpha \gamma \sigma_0] \\
d_2\rho(f_\alpha)(\Sigma_2) = d_2(C_0) = \alpha^{-1}d_2(D_0) = [\alpha^{-1}\sigma_0, \gamma \sigma_0]
\]

Recalling the construction of the previous section, order the summands and fix notation so that $\pi_1(P_1) \ast \cdots \ast \pi_1(P_r) = G_1 \ast \cdots \ast G_q \ast F_1 \ast \cdots \ast F_{r-q}$ where each $G_i$ is infinite and each $F_i$ is finite. As in the previous section, there is a $G \ast F$ action on the vector space $V = \oplus_{g \in G} \mathbb{R}^4$, on which each
Σ(i) acts as the zero endomorphism. Using this, we will construct a $\mathbb{Z}\pi_1(M)$-homomorphism from $\pi_2(M)$ to a certain $\mathbb{Z}\pi$-module $C_1(X; V)$. Let $X$ be the real line with the structure of a cell-complex having vertices the integers and 1-cells the segments of length 1 connecting two integer points. Denote by $s_n$ the 1-cell that runs from $n-1$ to $n$. Consider the module $C_1(X; V)$ of possibly infinite cellular 1-chains on $X$ with coefficients in $V$. A $\mathbb{Z}\pi$-module structure on $C_1(X; V)$ is defined on generators by $\gamma(v s_n) = v s_{n+1}$, and $g(v s_n) = g(v) s_n$ for $g \in \pi_1(P)$. Define a $\mathbb{Z}\pi$-homomorphism $\Psi: \pi_2(M) \to C_1(X; V)$ by sending $\sigma_0$ to $\sum_{\ell=1}^{\infty} v_1 s_{\ell}$, where $v_1 = ((1, 0, 0, 0), (0, 0, 0, 0), \ldots) \in V$, sending $\sigma_1$ to $-v_1 s_1$, and sending $\sigma_i$ to 0 for $i \geq 2$. Since $\Sigma(1)$ acts as the zero endomorphism of $V$, the element $\Sigma(1) \sigma_1$ is sent to 0. Also, $(1 - \gamma) \sigma_0 + \sigma_1 + \cdots + \sigma_r$ is sent to $\left( \sum_{\ell=1}^{\infty} v_1 s_{\ell} - \sum_{\ell=2}^{\infty} v_1 s_{\ell} \right) - v_1 s_1 = 0$.

Now let $W = X \times X$. This is $\mathbb{R}^2$ with a cell structure having vertices the integer lattice points, 1-cells the horizontal and vertical segments of length 1 connecting two integer lattice points, and 2-cells the squares of area 1 with integer lattice points for vertices. The notation $[v s_i, w s_j]$ will denote the 2-chain $[v, w](s_i \times s_j + s_j \times s_i)$ if $i \neq j$, and the 2-chain $[v, w](s_i \times s_i)$ if $i = j$. The coefficient $[v, w]$ is in $S(V)$. The bracket notation is extended linearly to chains. Define a $\mathbb{Z}$-homomorphism $S(\Psi)$ from $S(\pi_2(M))$ to the module of infinite 2-chains $C_2(W; S(V))$ by sending $[\sigma, \tau] \to [\Psi(\sigma), \Psi(\tau)]$.

**Proposition 8.1** Suppose that $\sum_{k=1}^{n} m_k f_{\alpha_k}$ is trivial in $\pi_1(\text{Diff}(M))$. Then $\sum_{k=1}^{n} m_k \alpha_k(v_1)$ is the zero element of $V$.

The proof of this proposition will require an easy lemma.

**Lemma 8.2** Let $W$ be a vector space over a field $K$ of characteristic $\neq 2$, and let $v$ be a nonzero element in $W$. If $[v, x] = 0$ in $S(W)$, then $x = 0$ in $W$.

**PROOF.** Fix a basis containing $v$. For $b$ in the basis, let $\pi_b: W \to K$ be the projection to the $b$-coordinate. For basis elements $a$ and $b$, there is a $K$-homomorphism $\psi_{a,b}: S(W) \to K$ defined by $\psi_{a,b}(y, z) = \pi_a(y) \pi_b(z) + \pi_a(z) \pi_b(y)$. Since $\psi_{v,b}(v, x) = \pi_b(x)$ if $b \neq v$ and $\psi_{v,v}(v, x) = 2\pi_v(x)$, the lemma follows. □

Next we will prove Proposition 8.1. If $\sum_{k=1}^{n} m_k f_{\alpha_k}$ is trivial in $\pi_1(\text{Diff}(M))$ then by the main diagram there exists $\eta \in \pi_2(M)$ so that
\[
[-\gamma \sigma_0, \eta] = \left[\sigma_0, \sum_{k=1}^{n} m_k \alpha_k \gamma \sigma_0\right]
\]
\[
[\sigma_0, \eta] = \left[\sum_{k=1}^{n} m_k \alpha_k^{-1} \sigma_0, \gamma \sigma_0\right]
\]

Applying \(S(\Psi)\) to both sides yields
\[
\begin{align*}
-\sum_{\ell=2}^{\infty} v_1 s_{\ell}, \Psi(\eta) &= \left[\sum_{\ell=1}^{\infty} v_1 s_{\ell}, \sum_{\ell=2}^{\infty} \left(\sum_{k=1}^{n} m_k \alpha_k(v_1)\right)s_{\ell}\right] \\
\sum_{\ell=1}^{\infty} v_1 s_{\ell}, \Psi(\eta) &= \left[\sum_{\ell=1}^{\infty} \left(\sum_{k=1}^{n} m_k \alpha_k^{-1}(v_1)\right)s_{\ell}, \sum_{\ell=2}^{\infty} v_1 s_{\ell}\right].
\end{align*}
\]

For a 2-chain \(c \in C_2(W; S(V))\), let \(\phi_{i,j}(c) \in S(V)\) denote the coefficient of the 2-cell \(s_i \times s_j\). Write \(\Psi(\eta)\) as \(\sum_{\ell=\infty}^{\infty} c_{\ell}s_{\ell}\). Applying \(\phi_{1,1}\) to the second of these two equations shows that \([v_1, c_1] = 0\) in \(S(V)\). Applying \(\phi_{1,2}\) to the first equation gives \(\sum_{k=1}^{n} m_k \alpha_k(v_1), v_1 = [v_1, c_1] = 0\), so by lemma 8.2 \(\sum_{k=1}^{n} m_k \alpha_k(v_1) = 0\) in \(V\). This completes the proof of proposition 8.1.

Now we will prove the remaining cases of the Main Theorem. If \(G \neq \{1\}\) then we may assume that \(\pi_1(P_1) = G_1\) is infinite. Taking the \(\alpha_k\) to be distinct elements of \(G_1\), the coordinate of \(\sum_{k=1}^{n} m_k \alpha_k(v_1)\) indexed by \(\alpha_k\) is \((m_k, 0, 0, 0)\).

By proposition 8.1, this shows \(\sum_{k=1}^{n} m_k f_{\alpha_k}\) is nontrivial in \(\pi_1(Diff(M))\) unless all \(m_k = 0\). Now suppose all \(\pi_1(P_i)\) are finite, but \(r \geq 2\). For each \(\pi_1(P_i)\) that is of order 2, we represent \(\pi_1(P_i)\) into \(SO(4)\) by sending the nontrivial element to \(\begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}\), so that \(\Sigma(\alpha)v_1 = 0\). Applying lemma 2.3 to a pair of elements \(\mu_1 \in \pi_1(P_1)\) and \(\mu_2 \in \pi_1(P_2)\) (where \(\mu_1 \neq -I\) in \(SO(4)\)), we may reselect the representation of \(\pi_1(P_2)\) so that in some coordinates \(\mu_1 \mu_2\) is of the form \(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\varphi} \end{pmatrix}\) where \(e^{i\theta}\) or \(e^{i\varphi}\) is a transcendental complex number. Note that since \(\Psi(\sigma_i) = 0\) for \(i \geq 2\), this does not destroy the fact that \(\Psi: \pi_3(M) \to C_1(X; V)\) is well-defined, although changing the representation of \(\pi_1(P_1)\) might do so. Putting \(\alpha_k = (\mu_1 \mu_2)^k\), \(\sum_{k=1}^{n} m_k \alpha_k(v_1)\) is nonzero unless all \(m_k = 0\) and proposition 8.1 implies the Main Theorem. This completes all cases of the proof of the Main Theorem.

**Theorem 8.3** Let \(P\) be a closed 3-manifold with finite fundamental group of order \(n\). Then \(\pi_1(Diff(S^2 \times S^1 \# P))\) contains a free abelian subgroup of rank \(n - 1\).
PROOF. Let \( M \) denote \( S^2 \times S^1 \# P \) and define \( V \) to be \( \mathbb{R}^n/((1, 1, \ldots, 1)) \), a vector space of dimension \( n-1 \). Let \( v_1 = (1, 0, 0, \ldots, 0) \) and let \( \pi_1(P) \) act on \( V \) by permuting the coordinates. Note that \( \Sigma(1)v_1 = (1, 1, \ldots, 1) = 0 \). A \( \mathbb{Z}\pi_1(M) \)-homomorphism \( \Psi: \pi_2(M) \to C_1(X; V) \) can be defined exactly as above. The proof of proposition 8.1 is unchanged. For any \( n-1 \) elements \( \alpha_1, \ldots, \alpha_{n-1} \) of \( F \), the vectors \( \{\alpha_i(v_1)\} \) are linearly independent so proposition 8.1 provides subgroups of \( \pi_1(\text{Diff}(M)) \) of rank \( n-1 \). □

Remark 8.4 The authors have examined imbeddings of manifolds similar to \( Y_n \) in \( S^2 \times S^1 \# P \) so that handles represent powers of \( \gamma \), in an effort to construct infinitely many linearly independent elements of \( \pi_1(\text{Diff}(S^2 \times S^1 \# P)) \) when \( \pi_1(P) \) is finite. These imbeddings have image disjoint from \( P \). Calculation of our invariants for these elements fails to show linear independence, but in fact one can see easily that such elements cannot generate a subgroup of rank more than 1. For consider the manifold \( S^2 \times S^1 \# D^3 \). Its diffeomorphism group is homotopy equivalent to \( \text{Diff}(S^2 \times S^1, x_0) \). Using the exact sequence for the evaluation fibration \( \text{Diff}(S^2 \times S^1, x_0) \to \text{Diff}(S^2 \times S^1) \to S^2 \times S^1 \), together with Hatcher’s [11] result that \( \text{Diff}(S^2 \times S^1) \) is homotopy equivalent to \( S^1 \times O(3) \times \Omega O(3) \), one finds that \( \pi_1(\text{Diff}(S^2 \times S^1, x_0)) \) has rank at most 1. Therefore any set of isotopies of \( S^2 \times S^1 \# P \) which fix all points in \( P \) can generate at most a rank 1 subgroup of \( \pi_1(\text{Diff}(S^2 \times S^1 \# P)) \).

9 The case of the connected sum of two irreducible 3-manifolds

Throughout this section, we assume that \( M \) is a connected sum of two irreducible 3-manifolds. The summands meet along a 2-sphere \( S \) in \( M \), and it is easy to check that up to isotopy \( S \) is the unique 2-sphere imbedded in \( M \) that does not bound a 3-ball.

Theorem 9.1 Let \( M \) be a connected sum of two irreducible 3-manifolds and let \( S \subset M \) be a 2-sphere that does not bound a 3-ball. Then for all \( k \geq 0 \), \( \pi_k(\text{Diff}(M), \text{Diff}(M, S)) \) is trivial.

This theorem is given as the final remark in [11]. Since we have found its verification to be more than a routine exercise, we include details of its deduction from Hatcher’s main results here as lemma 9.3 and proposition 9.4. The case of \( k=0 \) is simply the fact that \( S \) is unique up to isotopy. The case of \( k=1 \) is one of the main results of the dissertation of B. Jahren [23]. The cases \( k \geq 1 \) are immediate from proposition 9.4 below. Using theorem 9.1, standard fibrations of mappings spaces yield the following statement.

Theorem 9.2 Let \( M = M_1 \# M_2 \) be a connected sum of two compact \( \mathbb{P}^2 \)-irreducible 3-manifolds. Then for \( k \geq 0 \), \( \pi_k(\text{Diff}(M)) \) is finitely generated.
if and only if \( \pi_k(\text{Diff}(M_1)) \) and \( \pi_k(\text{Diff}(M_2)) \) are finitely generated.

The proof of theorem 9.2 will be given below after the proofs of lemma 9.3 and proposition 9.4.

Throughout this section, when \( g: X \times I^k \to Y \) is a parameterized family of imbeddings or diffeomorphisms, the notation \( g_t \) will denote the function from \( X \) to \( Y \) defined by \( g_t(x) = g(x, t) \). Although the interval \([-1, 1]\) will be used, the symbol \( I \) will always denote the interval \([0, 1]\). A diffeomorphism of \( S^2 \times [-1, 1] \) is said to take levels to levels if each subspace \( S^2 \times \{t_1\} \) is carried to some \( S^2 \times \{t_2\} \). It is called level preserving when it carries each \( S^2 \times \{t\} \) to itself.

**Lemma 9.3** Let \( S^2 \times [-1, 1] \) be imbedded in the interior of a 3-manifold \( M \). Let \( S \) be a copy of \( S^2 \) and let \( R \) be a nonempty subset of \([-1, 1]\). Let \( f: S \times I^k \to M - S^2 \times R \) be a \( k \)-parameter family of imbeddings, where \( k \geq 1 \). Suppose that for some \( t_0 \in \partial I^k \), \( f_{t_0}(S) = S^2 \times \{0\} \).

(i) There exists a \((k + 1)\)-parameter family \( Q: (S \times I^k) \times I \to M - S^2 \times R \) such that \( Q_0 = f \), \( Q_{(t,1)}(S) = S^2 \times \{0\} \) for all \( t \in I^k \), and \( Q_{(t_0, a)} = f_{t_0} \) for all \( s \in I \).

(ii) If in addition \( f_{1}(S)=S^2 \times \{0\} \) for all \( t \in \partial I^k \), then \( Q \) may be selected so that \( Q_{(t,s)} = f_t \) for all \( t \in \partial I^k \) and all \( s \in I \).

**Proof.** For (i), simply fix a deformation retraction \( j: I^k \times I \to I^k \) such that \( j_1(I^k) = t_0 \), and define \( Q_{t,s} = f_{j_{(t,s)}} \). For (ii), select notation so that \( 1 \in R \). If \(-1\) is also in \( R \), let \( M_0 = M - S^2 \times \{-1\} \), otherwise let \( M_0 = M \). Let \( g: S \to S^2 \) be the homeomorphism obtained from \( f_{t_0}: S \to M \) by composition with the inverse of the inclusion \( S^2 \to S^2 \times \{0\} \subset S^2 \times [-1, 1] \subset M \). Define a \( k \)-parameter family of imbeddings \( J: (S \times I) \times I^k \to M_0 \) as follows. Define \( J: S \times \partial I \times I^k \cup S \times I \times \{t_0\} \to M \) by \( J(x, 0, t) = f(x, t) \), \( J(x, 1, t) = g(x, 1) \), and \( J(x, s, t_0) = (g(x), s) \). Using the Parameterized Isotopy Extension Theorem (i.e. the fact that the map of spaces of smooth imbeddings \( \text{Imb}(S \times I, M_0) \to \text{Imb}(S \times \partial I, M_0) \) determined by restriction is a fibration [40]), \( J \) extends to a \( k \)-parameter family of imbeddings. Since \( J_{t_0}(S \times I) = S^2 \times I \), \( J_t(S \times I) = S^2 \times I \) for all \( t \in \partial I^k \). By assertion b) in [11], which is equivalent to the Smale Conjecture, the space of diffeomorphisms of \( S^2 \times I \) deformation retracts onto the subspace of diffeomorphisms of \( S^2 \times I \) taking levels to levels, which deformation retracts onto the subspace of level preserving diffeomorphisms of \( S^2 \times I \). This implies that we may choose \( J \) so that \( J_t: S \times I \to S^2 \times I \) is level preserving for all \( t \in \partial I^k \).

For \( s \in I \) define a 1-parameter family of diffeomorphisms \( h_s: M \to M \) such that \( h_0 \) is the identity, \( h_s(x, s) = (x, 0) \) for \((x, s) \in S^2 \times I \). \( h_s \) has support in a small neighborhood of \( S^2 \times I \) (in particular, it is the identity on \( S^2 \times \{-1\} \),
and therefore $h_s(M_0) = M_0$, and $h_s$ takes levels of $S^2 \times [-1,1]$ to levels. Now define a $(k+1)$-parameter family of imbeddings $K: S \times I^k \times I \to M_0$ by $K(x,t,s) = h_{0} \circ J(x,s,t)$. Then $K(x,t,0) = h_0 \circ J(x,0,0) = f(x,t)$ and $K(x,t,1) = h_1 \circ J(x,1,1) = h_1(g(x),1) = (g(x),0) = f(x,t_0)$.

Since $J_{s,t}(S) \subset M_0$, and $h_s(M_0) = M_0$, we have $K_{t,s}(S) \subset M - S^2 \times \{-1\}$ if $-1 \in R$. Also, $J_{s,t}(S)$ meets $S^2 \times \{1\}$ exactly when $s = 1$, and for all $s$, $J_{s,t}(S)$ is disjoint from some collar neighborhood of $S^2 \times \{1\}$ in $M - S^2 \times (-1,1)$. Provided that the support of $h_s$ was chosen to lie in the union of $S^2 \times (-1,1)$ and this neighborhood, $h_s \circ J_{s,t}(S)$ will be disjoint from $S^2 \times \{1\}$. Therefore $K_{t,s}(S) \subset M - S^2 \times R$.

For all $t \in \partial I^k$, $K_{t,s}(S) = h_s \circ J_{s,t}(S) = h_s(S^2 \times \{s\}) = S^2 \times \{0\}$ for all $s \in I$. Define a $(k+1)$-parameter family $L: S \times I^k \times I \to S^2 \times \{0\}$ as follows. Define $L: S \times \partial I^k \times I \cup S \times I^k \times \{0\} \to S^2 \times \{0\}$ by $L(x,t,s) = K(x,t,1-s)$. Then, regard $L$ as a map from $\partial I^k \times I \cup I \times \{0\}$ to $\Diff(S^2)$ and extend to $I^k \times I$ using the Homotopy Extension property. Define $Q: (S \times I^k) \times I \to M$ by

$$Q(x,t,s) = \begin{cases} K(x,t,2s), & 0 \leq s \leq 1/2 \\ L(x,t,2s-1), & 1/2 \leq s \leq 1. \end{cases}$$

Note that $Q(x,t,0) = f(x,t)$, $Q_{t,0}(S) = S^2 \times \{0\}$, and for all $t \in \partial I^k$ and $s \in I$, $Q_{t,s}(S) = S^2 \times \{0\}$ and $Q(x,t,0) = Q(x,t,1)$. Also, the image of $Q$ is disjoint from $S^2 \times R$. By a deformation supported in a neighborhood of $S \times \partial I^k$, we may modify $Q$ so that $Q(x,t,s) = Q(x,t,0)$ for all $t \in \partial I^k$ and for all $s \in I$.

**Proposition 9.4** Let $M$ be a connected sum of two irreducible 3-manifolds and let $S^2 \subset M$ be a 2-plane that does not bound a 3-ball. Let $S$ be a copy of $S^2$ and suppose $f: S \times I^k \to M$ is a $k$-parameter family of imbeddings such that $f_t(S) = S^2 \times \{0\}$ for all $t \in \partial I^k$. Then there exists a $(k+1)$-parameter family of imbeddings $Q: (S \times I^k) \times I \to M$ such that $Q_0 = f$, $Q_{(t,1)}(S) = S^2 \times \{0\}$ for all $t \in I^k$, and $Q_{(t,s)} = f_t$ for all $s \in I$ and $t \in \partial I^k$.

**Proof.** Let $S^2 \times [-1,1]$ be an imbedded regular neighborhood of $S^2 = S^2 \times \{0\}$. By [11] we may assume that for any $t \in I^k$ there exists $x_t \in [0,1]$ such that $f_t(S^2)$ does not meet $S^2 \times \{x_t\}$. Let $p: M \to [0,1]$ be the map defined by $p(S^2 \times \{i\}) = i$, and each component of $M - S^2 \times [-1,1]$ maps appropriately to either $-1$ or 1. Let $B_{-1} = \{ t \in I^k \mid -1 \notin pf_t(S^2) \}$ and let $B_1 = \{ t \in I^k \mid 1 \notin pf_t(S^2) \}$. Note that both $B_{-1}$ and $B_1$ are open sets. Furthermore since $x_t \notin pf_t(S^2)$, $B_{-1} \cup B_1 = I^k$. Using a Lebesgue number for this covering we may triangulate $I^k$ so that every simplex is contained in $B_{-1}$ or $B_1$. We will construct a sequence of deformations of $f$, fixed over $\partial I^k$ and each supported in a neighborhood of a simplex of $I^k$. These will be constructed...
first on the 1-simplices in a tree, then over the remaining 1-simplices, and then successively over higher dimensional simplices.

In the 1-skeleton of $I^k$, let $T'$ be a tree maximal among the trees disjoint from $\partial I^k$, and obtain a tree $T$ by adding to $T'$ a 1-simplex $\sigma_0$ that connects $T'$ to $\partial I^k$. Working inductively through the 1-simplices in $T$, starting with $\sigma_0$, we may assume that one endpoint $t_0$ of our 1-simplex $\sigma$ lies in the portion of the maximal tree where the deformation has already been carried out. We will use $f$ to denote the result of the deformation that has already occurred; in particular, $f_{t_0}(S) = S^2 \times \{0\}$ and this will be held fixed during the deformation. Let $R = \{i \in \{-1, 1\} | \partial_i \in B_i\}$. Consider the 1-parameter family $f|_{S^2 \times \sigma}: S^2 \times \sigma \to M - S^2 \times R$. By lemma 9.3(i) there exists a deformation $F_t: S^2 \times \sigma \to M - S^2 \times R$, which is relative to $S^2 \times \{t_0\}$, such that $F_0 = f$ and $(F_t)_{t_0}(S) = S^2 \times \{0\}$ for all $t \in \sigma$. Let $t_1$ be the endpoint of $\sigma$ other than $t_0$, and let $V$ be the star of $t_1$ in $I^k$. Define $F_t|_{S^2 \times (I^k - V)} = f|_{S^2 \times (I^k - V)}$ for all $s \in I$, and use the Homotopy Extension property to extend $F_s$ over all of $V$. Note that if $\tau$ is a simplex in $V$, and $\tau \subset B_i$, then $\sigma \subset B_i$ and the isotopy $F_s$ obtained from lemma 9.3(i) carries $S \times \sigma$ into $M - S^2 \times \{t\}$. Therefore the homotopy extension may be selected so that $\tau$ remains in $B_i$. Denote the improved map $F_1$ by $f$ again. This shows how to deform $f$ over all the 1-simplices in $T$ so that $f_t(S) = S^2 \times \{0\}$ for any $t$ in this tree. The same procedure can then be used to deform $f$ over the remaining 1-simplices by using lemma 9.3(ii) in place of lemma 9.3(i), and the union of the simplices that meet the interior of $\sigma$ as $V$. The result is a new $f$ so that $f_t(S) = S^2 \times \{0\}$ for any $t$ in a 1-simplex. Applying lemma 9.3(ii) over sequentially higher dimensional simplices completes the proof of proposition 9.4.

**Proof of theorem 9.2** Let $S$ be an imbedded sphere in $M$ which does not bound a 3-ball. Let $\text{Diff}(M, S)$ denote the space of diffeomorphisms which leave $S$ invariant, and let $\text{Diff}(M \text{ rel } S)$ denote the space of diffeomorphisms of $M$ that restrict to the identity on $S$. A subscript 0, as in $\text{Diff}_0(M)$, indicates the connected component of the identity map. (For all mapping spaces, the identity map is the basepoint.)

Recall from [43] that $\text{Diff}_0(S^2) \simeq \text{SO}(3)$. Consequently, $\pi_1(\text{Diff}(S)) \cong \mathbb{Z}/2$ and for $k \geq 2$, $\pi_k(\text{Diff}(S)) \cong \pi_k(\text{SO}(3)) \cong \pi_k(S^3)$ which is finite. By theorem 9.1 and the exact sequence for the pair $(\text{Diff}_0(M), \text{Diff}_0(M, S))$, it follows that $\pi_k(\text{Diff}_0(M)) \simeq \pi_k(\text{Diff}_0(M, S))$. There is a fibration $\text{Diff}_1(M \text{ rel } S) \hookrightarrow \text{Diff}_0(M, S) \rightarrow \text{Diff}_0(S)$ where the last map is given by restriction and the fiber $\text{Diff}_1(M \text{ rel } S)$ is $\text{Diff}(M \text{ rel } S) \cap \text{Diff}_0(M, S)$. It yields the following exact sequences for $k \geq 1$:

$$
\pi_{k+1}(\text{Diff}(S)) \rightarrow \pi_k(\text{Diff}(M \text{ rel } S)) \rightarrow \pi_k(\text{Diff}(M, S)) \rightarrow \pi_k((\text{Diff}(S))
$$
Therefore $\pi_k(\text{Diff}(M))$ is finitely generated if and only if $\pi_k(\text{Diff}(M \text{ rel } S))$ is finitely generated.

Let $\text{Diff}_{\text{abd}}(X \text{ rel } A)$ denote the subspace of $\text{Diff}_0(X \text{ rel } A)$ consisting of the diffeomorphisms that are the identity on some neighborhood of $A$. When $A$ is a submanifold, standard techniques of differential topology show that $\text{Diff}_{\text{abd}}(X \text{ rel } A) \simeq \text{Diff}_0(X \text{ rel } A)$.

If $M_1'$ and $M_2'$ are the closures of the components of $M - S$, then we have $\text{Diff}_0(M \text{ rel } S) \simeq \text{Diff}_{\text{abd}}(M \text{ rel } S) = \text{Diff}_{\text{abd}}(M_1' \text{ rel } S) \times \text{Diff}_{\text{abd}}(M_2' \text{ rel } S)$.

Let $D$ be a 3-ball in $M_i$ whose boundary is $S$. We have $\text{Diff}_{\text{abd}}(M_i \text{ rel } D) \simeq \text{Diff}_0(M_i \text{ rel } D)$. The exact sequence of the fibration $\text{Diff}(M_i \text{ rel } D) \rightarrow \text{Diff}(M_i, D) \rightarrow \text{Diff}(D)$ is

$$\pi_{k+1}(\text{Diff}(D)) \rightarrow \pi_k(\text{Diff}(M_i \text{ rel } D)) \rightarrow \pi_k(\text{Diff}(M_i, D)) \rightarrow \pi_k(\text{Diff}(D)).$$

From [9], $\text{Diff}_0(D^3) \simeq \text{SO}(3)$, so $\pi_k(\text{Diff}(M_i'))$ is finitely generated if and only if $\pi_k(\text{Diff}(M_i, D))$ is finitely generated. Letting $x_i$ be a basepoint in $E$, we have $\pi_k(\text{Diff}(M_i, D)) \simeq \pi_k(\text{Diff}(M_i, x_i))$. The fibration $\text{Diff}(M_i, x_i) \rightarrow \text{Diff}(M_i) \rightarrow \text{int}(M_i)$ yields the exact sequence

$$\pi_{k+1}(M_i) \rightarrow \pi_k(\text{Diff}(M_i, x_i)) \rightarrow \pi_k(\text{Diff}(M_i)) \rightarrow \pi_k(M_i).$$

Since each $M_i$ is $\mathbb{P}^2$-irreducible, its universal cover is either contractible or is a homotopy 3-sphere, hence all its homotopy groups are finitely generated. So $\pi_k(\text{Diff}(M_i, x_i))$ is finitely generated if and only if $\pi_k(\text{Diff}(M_i))$ is finitely generated. □

10 Manifolds with 2-sphere boundary components

In this section we examine 3-manifolds with 2-sphere boundary components. If $N$ is a 3-manifold, let $\hat{N}$ denote the manifold obtained by filling in all 2-sphere boundary components of $N$ with 3-balls, and let $\mathcal{P}(N)$ denote the Poincaré associate of $N$, which is the 3-manifold obtained by replacing each homotopy 3-sphere prime summand with an $S^3$ summand. Our main result reduces the question of finite generation of $\pi_1(\text{Diff}(M))$ when $M$ has a 2-sphere boundary component to the corresponding question for prime manifolds.

**Theorem 10.1** Let $N$ be a 3-manifold which is compact and orientable. Then $\pi_1(\text{Diff}(N \# D^3))$ is finitely generated if and only if one of the following three holds.
We will organize the proof of theorem 10.1 into a sequence of lemmas. The next fact is well-known.

**Lemma 10.2** Let $N$ be a closed 3-manifold with fundamental group of order 2. Then $N$ is homotopy equivalent to $\mathbb{RP}^3$.

**PROOF.** There is a map from $\mathbb{RP}^2$ to $N$ which induces an isomorphism on fundamental groups. Since $\pi_2(N) = 0$, it extends to a map $f: \mathbb{RP}^3 \to N$. Letting $\gamma$ be a generator of $H^1(N; \mathbb{Z}/2) \cong \mathbb{Z}/2$, $f^*(\gamma)$ generates $H^1(\mathbb{RP}^3; \mathbb{Z}/2)$. Since $f^*(\gamma^3) = f^*(\gamma)^3$ generates $H^3(\mathbb{RP}^3; \mathbb{Z}/2)$, it follows that $\gamma^3$ generates $H^3(N; \mathbb{Z}/2)$ and $f^*: H^3(N; \mathbb{Z}/2) \to H^3(\mathbb{RP}^3; \mathbb{Z}/2)$ is an isomorphism. Therefore $f$ has odd degree. By Theorem IV of [39], $\mathbb{RP}^3$ and $N$ must be homotopy equivalent. (One can also see this directly: Since the universal cover of $N$ is a homotopy 3-sphere, and is a 2-fold cover of $N$, one can use a modification in a 3-ball to change the degree of $f$ by any even integer, without altering it on the 2-skeleton. Therefore there is a degree 1 map inducing an isomorphism of fundamental groups. Its lift to the universal covers has degree 1, so induces an isomorphism on all homotopy groups.) \(\Box\)

From now on, we use $\tilde{N}$ to denote the universal cover of $N$.

**Lemma 10.3** Let $N$ be an orientable 3-manifold. If $\pi_1(N)$ is infinite, and $P(N)$ is prime and not $S^2 \times S^1$, then $\tilde{N}$ is contractible. If $N$ is closed and has finite fundamental group, then $\tilde{N}$ is a homotopy 3-sphere. In all other cases, $\tilde{N}$ is homotopy equivalent to a one-point union of a collection of 2-spheres.

**PROOF.** The first two cases are well-known exercises in algebraic topology. For the remaining case, $\tilde{N}$ is not closed so $H_3(\tilde{N}) = 0$. Since $H_1(\tilde{N}) = 0$, $H_2(\tilde{N}) \cong \pi_2(\tilde{N})$, which is free abelian by a theorem of Specker [44] (see pp. 147-149 of [15]). If $P(N) = S^2 \times S^1$, then $\tilde{N} \cong S^2 \times \mathbb{R} \cong S^3$. Otherwise, $P(N)$ is not prime, so $N$ either has a 2-sphere boundary component or contains a separating 2-sphere with neither complementary component simply-connected; in either case its lift to $\tilde{N}$ cannot bound so $H_2(\tilde{N}) \neq 0$. There is a map from a 1-point union of 2-spheres to $\tilde{N}$ which induces isomorphisms on the fundamental group and all homology groups, so it is a homotopy equivalence.

**Lemma 10.4** Suppose $N$ is a compact orientable 3-manifold for which some nonzero element of $H_2(\tilde{N}; \mathbb{Z}/2)$ is fixed by the action of infinitely many ele-
ments of $\pi_1(N)$. Then $N$ is homotopy equivalent to $S^2 \times S^1$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$.

**PROOF.** We will first show that $N$ is closed and $\pi_1(N)$ has two ends (a useful reference on ends is [4]). Let $F$ be a compact connected imbedded 2-manifold in the interior of $\widetilde{N}$ representing the fixed element, so for infinitely many $\gamma_i$, $\gamma(F)$ is homologous to $F$. Since $\widetilde{N}$ is simply-connected, $F$ must separate $\widetilde{N}$ into two components, say $N_1$ and $N_2$. Since $\pi_1(N)$ is infinite and acts properly discontinuously on $\widetilde{N}$, there are infinitely many disjoint translates $F_1$, $F_2$, $\ldots$ of $F$ so that for each $i$, $F$ and $F_i$ cobound a compact submanifold $W_i$ of $M$. We may assume that these all lie in $N_2$. If $\gamma_i$ carries $F$ to $F_i$, then either $\gamma_i^{-1}(F)$ or $\gamma_i^{-1}(F_i)$ lies in $N_1$. Therefore we can find a sequence $F_{-i}$ of translates of $F$ in $N_1$ such that $F_{-i}$ and $F_i$ cobound a compact submanifold $V_i$ which lies in $N_1$. Therefore $F_{-i}$ and $F_i$ cobound a compact submanifold $U_i = V_i \cup W_i$. Since each $U_i$ can meet only finitely many $F_j$, we may pass to a subsequence so that $U_i$ is contained in the interior of $U_{i+1}$. The union of the $U_i$ is a submanifold with no frontier, so equals all of $\widetilde{N}$, showing that $\widetilde{N}$ has two ends. Therefore $\pi_1(N)$ has two ends, so contains an infinite cyclic subgroup of finite index. Since $\partial U_i = F_{-i} \cup F_i$, $\widetilde{N}$ has no boundary so $N$ is closed.

Suppose first that $\mathcal{P}(N)$ is prime, then $\pi_1(N)$ is a torsionfree finite extension of an infinite cyclic group, so is infinite cyclic. This follows from theorem 10.7 of [14] (or from the group theoretic fact that a torsionfree finite extension of a free group must be free). Since $N$ is closed, $\mathcal{P}(N) = S^2 \times S^1$. Now suppose $\mathcal{P}(N)$ is not prime. Then $\pi_1(N)$ is a free product. The only free product with two ends is $\mathbb{Z}/2 * \mathbb{Z}/2$, so $\mathcal{P}(N) = N_1 \# N_2$ where each $\pi_1(N_i)$ has order 2. By lemma 10.2, each $N_i$ is homotopy equivalent to $\mathbb{RP}^3$.

**Lemma 10.5** Let $N$ be a 3-manifold which is compact and orientable. Then $\pi_2(N)$ is finitely generated (as an abelian group) if and only if either $\mathcal{P}(N)$ is prime, or $\pi_1(N)$ is finite, or $N$ is homotopy equivalent to $\mathbb{RP}^3 \# 2 \mathbb{RP}^3$.

**PROOF.** If $N$ is one of these three kinds, then $\widetilde{N}$ is either contractible, or a punctured homotopy 3-sphere, or a homotopy $S^2 \times \mathbb{R}$. Conversely, assume that $\pi_1(N)$ is infinite and $\pi_2(N)$ is finitely generated. If $\pi_2(N)$ is zero, then from lemma 10.3, $\widetilde{N}$ is contractible and $\mathcal{P}(N)$ is prime. If $\pi_2(N)$ is nonzero and finitely generated, then $H_2(N; \mathbb{Z}/2)$ is finite, so every element is fixed by infinitely many elements of $\pi_1(N)$. By lemma 10.4, this implies that $\mathcal{P}(N)$ is either $S^2 \times S^1$, hence prime, or is homotopy equivalent to $\mathbb{RP}^3 \# 2 \mathbb{RP}^3$.

**Lemma 10.6** Let $N$ be a compact orientable 3-manifold and let $x_0$ be a point in $N$. Then there is an exact sequence

$$\pi_2(N) \rightarrow \pi_1(\text{Diff}(N, x_0)) \rightarrow \pi_1(\text{Diff}(N)) \rightarrow \text{center}(\pi_1(N)).$$

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Suppose that $\pi_1(N)$ is infinite, and $N$ is not homotopy equivalent to $S^2 \times S^1$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$. Then the homomorphism from $\pi_2(N)$ to $\pi_1(\text{Diff}(N,x_0))$ is injective.

**PROOF.** Evaluation at $x_0$ gives a fibration $\text{Diff}(N) \to \text{int}(N)$ with fiber $\text{Diff}(N,x_0)$, so there is an exact sequence

$$\pi_2(\text{Diff}(N)) \to \pi_2(N) \to \pi_1(\text{Diff}(N,x_0)) \to \pi_1(\text{Diff}(N)) \to \pi_1(N).$$

The next part of the argument is essentially the same as Lemma 2.A.5 of [31]. Consider a parameterized family of diffeomorphisms $g: N \times I^k \to N$ representing an element of $\pi_k(\text{Diff}(N))$, so $g_t = 1_N$ for all $t \in \partial I^k$. The restriction of $g$ to $x_0 \times I^k$ represents an element of $\pi_k(N)$. Suppose $k = 1$. Let $\alpha_t$ be a path that runs along $x_0 \times I$ from $x_0 \times \{0\}$ to $x_0 \times \{t\}$, where $t \in I^*$, and let $\beta$ be a loop in $N$. Then the restrictions of $g$ to $\alpha_t \times \{0\}$ defines a homotopy from $g\#(\alpha_1)\beta g\#(\alpha_1)^{-1}$ to $\beta$, showing that $g\#(\alpha_1)$ is central. Therefore the map from $\pi_1(\text{Diff}(N))$ to $\pi_1(N)$ has image in the center of $\pi_1(N)$. Now suppose $k \geq 2$. Let $h: N \times I^k \to N$ be the constant family for which $h_t = 1_N$ for all $t \in I^k$. Then the difference element $d_k(g,h)$ lies in $C^k(N \times I^k, N \times \partial I^k; \pi_k(N))$, the $k$-cochains with local coefficients in $\pi_k(N)$. The coboundary $\delta d_k(g,h)$ is zero since both $g$ and $h$ extend to the $(k+1)$-skeleton. We will show that $\delta d_k(g,h) = 0$ implies that $d_k(g,h)(x_0 \times I^k) = 0$. Let $\gamma$ be any loop in $N$ based at $x_0$. It determines a ($k+1$)-dimensional class $\gamma \times I^k$ in $C_{k+1}(N \times I^k, N \times \partial I^k; \pi_k(N))$, and we have

$$0 = \delta d_k(g,h)(\gamma \times I^k) = \gamma d_k(g,h)(x_0 \times I^k) - d_k(g,h)(x_0 \times I^k),$$

showing that $d_k(g,h)(x_0 \times I^k)$ is in the subgroup of $\pi_k(N,x_0)$ fixed under the action of $\pi_1(N)$. Now suppose that $k = 2$. If a nonzero element of $\pi_2(N)$ is fixed under the action of $\pi_1(N)$, then since $H_2(\tilde{N}) \cong \pi_2(N)$ is free abelian, there exists a nonzero element of $H_2(\tilde{N}; \mathbb{Z}/2)$ fixed under the action of $\pi_1(N)$. Therefore lemma 10.4 shows that $\pi_1(N)$ is finite or $N$ is homotopy equivalent to $S^2 \times S^1$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$. Apart from these cases, $d_2(g,h)(x_0 \times I^2) = 0$, so the homomorphism $\pi_2(\text{Diff}(N)) \to \pi_2(N)$ is zero.

**Proposition 10.7** Let $N$ be a compact orientable 3-manifold with a basepoint $x_0$ in its interior. Then for all $k \geq 1$, $\pi_k(\text{Diff}(N \# D^3))$ is finitely generated if and only if $\pi_k(\text{Diff}(N,x_0))$ is finitely generated.

**PROOF.** Let $S = \partial D^3$ and regard $N$ as obtained from $N \# D^3$ by filling in $S$ with a 3-ball $D$; we may assume that $x_0$ is the center point of $D$. Since $k \geq 1$,
$\pi_k(\text{Diff}(N \# D^3)) \cong \pi_k(\text{Diff}(N \# D^3, S))$. As in the proof of theorem 9.2, the exact sequence

$$
\pi_{k+1}(\text{Diff}(S)) \to \pi_k(\text{Diff}(N \# D^3 \text{ rel } S)) \to \pi_k(\text{Diff}(N \# D^3, S)) \to \pi_k(\text{Diff}(S))
$$

shows that $\pi_k(\text{Diff}(N \# D^3, S))$ is finitely generated if and only if the same is true for $\pi_k(\text{Diff}(N \# D^3 \text{ rel } S))$. Using the notation of section 9, we have

$$
\text{Diff}(N \# D^3 \text{ rel } S) \simeq \text{Diff}_{\text{nbhd}}(N \# D^3 \text{ rel } S) \simeq \text{Diff}_{\text{nbhd}}(N \text{ rel } D) \simeq \text{Diff}(N \text{ rel } D)
$$

The Smale Conjecture and the exact sequence

$$
\pi_{k+1}(\text{Diff}(D)) \to \pi_k(\text{Diff}(N \text{ rel } D)) \to \pi_k(\text{Diff}(N, D)) \to \pi_k(\text{Diff}(D))
$$

show that $\pi_k(\text{Diff}(N \text{ rel } D))$ is finitely generated if and only if $\pi_k(\text{Diff}(N, D))$ is finitely generated, and $\pi_k(\text{Diff}(N, D)) \cong \pi_k(\text{Diff}(N, x_0))$.

We can now complete the proof of theorem 10.1. By [38], the center of the fundamental group of a compact 3-manifold is always finitely generated. If either $\mathcal{P}(N)$ is prime or $N \simeq \mathbb{RP}^3 \# \mathbb{RP}^3$, then $\pi_2(N)$ is finitely generated so lemma 10.6 implies that $\pi_1(\text{Diff}(N, x_0))$ is finitely generated if and only if $\pi_1(\text{Diff}(N))$ is. If $\pi_1(N)$ is finite, then again by lemma 10.6 $\pi_1(\text{Diff}(N, x_0))$ is finitely generated if and only if $\pi_1(\text{Diff}(N))$ is, and by inducting on the number of 2-sphere boundary components of $N$, this will be finitely generated if and only if $\pi_1(\text{Diff}(\breve{N}))$ is. In all remaining cases, lemmas 10.5 and 10.6 imply that $\pi_2(N)$ is an infinitely generated subgroup of $\pi_1(\text{Diff}(N, x_0))$.

References


[38] G. Mess, The Seifert Conjecture and groups which are coarse quasiisometric to planes, preprint.


