Math 2433 Homework #7 and #8
Solutions

Section 12.9.

Find the power series representation of the functions below and find the interval of convergence of the series.

8. \( f(x) = \sum \frac{x}{4x+1} \).

**Solution:** We have
\[
\frac{x}{4x+1} = x \left( \frac{1}{1+4x} \right).
\]
The power series representation of \( \frac{1}{1+4x} \) is
\[
\sum_{n=0}^{\infty} (-4x)^n = \sum_{n=0}^{\infty} (-1)^n 4^n x^n
\]
so the power series representation of \( f \) is
\[
\sum_{n=0}^{\infty} (-1)^n 4^n x^{n+1}.
\]
The series is convergent for \( | -4x | < 1 \), that is, \( |x| < \frac{1}{4} \). The interval of convergence is \((-\frac{1}{4}, \frac{1}{4})\).

10. \( f(x) = \frac{x^2}{a^3-x^3} \).

**Solution:** Here \( a \) is a constant. We again have
\[
\frac{x^2}{a^3-x^3} = x^2 \left( \frac{1}{a^3-x^3} \right)
\]
so we need only find the power series representation of \( \frac{1}{a^3-x^3} \). To do that we need to write the denominator as \( 1 - \text{something} \). We do this as follows:
\[
\frac{1}{a^3-x^3} = \frac{1}{a^3} \left( \frac{1}{1 - \left( \frac{x}{a} \right)^3} \right).
\]
We know that the power series representation of \( \frac{1}{1 -(\frac{x}{a})^3} \) is
\[
\sum_{n=0}^{\infty} \left( \frac{x}{a} \right)^{3n} = \sum_{n=0}^{\infty} \frac{x^{3n}}{a^{3n}}.
\]
This series converges for \[ \left| \frac{x}{a} \right| < 1. \]
That is, when \(|x| < |a|\). So the interval of convergence is \((-|a|, |a|)\). The power series representation of the function \(f\) is then
\[
\frac{x^2}{a^3} \sum_{n=0}^{\infty} \frac{x^{3n}}{a^{3n}} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{a^{3n+3}}
\]
and the interval of convergence is \((-|a|, |a|)\).

Express the given function as a sum of power series by first using partial fraction decomposition.

12. \(\frac{7x-1}{3x^2+2x-1}\).

**Solution:** We first factorize the denominator to get
\[3x^2 + 2x - 1 = (1 + x)(3x - 1).\]
Now, let
\[\frac{7x-1}{3x^2+2x-1} = \frac{A}{x+1} + \frac{B}{3x-1}\]
where \(A\) and \(B\) are constants to be determined. We must have

\[7x - 1 = A(3x - 1) + B(x + 1) = (3A + B)x + (B - A)\]

which implies that \(B - A = -1\) and \(3A + B = 7\). Solving for \(A\) and \(B\) gives us \(A = 2\) and \(B = 1\). The partial fraction decomposition is thus

\[\frac{7x-1}{3x^2+2x-1} = \frac{2}{x+1} + \frac{1}{3x-1}.\]

The power series representation of \(\frac{2}{x+1} = \frac{2}{1+x}\) is

\[2 \sum (-x)^n = \sum (-1)^n 2x^n\]
and the series converges for \(|-x| < 1\). The interval of convergence is \((-1, 1)\). The power series of the second fraction \(\frac{1}{3x-1} = \frac{1}{1-3x}\) is

\[-\sum (3x)^n = -\sum 3^n x^n\]
and the series converges for $|3x| < 1$ so the interval of convergence is $(-\frac{1}{3}, \frac{1}{3})$. The power series representation of $f$ is therefore

$$\sum (-1)^n 2x^n - \sum 3^n x^n = \sum ((-1)^n 2 - 3^n) x^n$$

and the interval of convergence is the smaller of the intervals of convergence of the two individual fractions. So the interval of convergence is $(-\frac{1}{3}, \frac{1}{3})$.

13. (a) Use differentiation to find a power series representation of

$$f(x) = \frac{1}{(1 + x)^2}.$$ 

What is the radius of convergence?

(b) Use Part (a) to find the power series representation of

$$\frac{1}{(1 + x)^3}.$$ 

(c) Use Part (b) to find the power series representation of the function

$$f(x) = \frac{x^2}{(1 + x)^3}.$$ 

Solution: (a) We notice that

$$\frac{1}{(1 + x)^2} = -\frac{d}{dx} \left( \frac{1}{1 + x} \right)$$

We know that the power series representation of $\frac{1}{1+x}$ is

$$\sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

which has radius of convergence $R = 1$. Therefore the power series representation of $f$ is

$$-\frac{d}{dx} \left( \sum_{n=0}^{\infty} (-1)^n x^n \right) = -\sum_{n=1}^{\infty} (-1)^n n x^{n-1} = \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1}.$$ 

The radius of convergence remains $R = 1$.

(b) We again notice that

$$\frac{1}{(1 + x)^3} = -\frac{1}{2} \frac{d}{dx} \left( \frac{1}{(1 + x)^2} \right).$$
We already know the power series of \(\sum \frac{1}{(1+x)^2}\) from (a) so the power series of \(\frac{1}{(1+x)^3}\) is
\[-\frac{1}{2} \frac{d}{dx} \left( \sum_{n=1}^{\infty} (-1)^{n+1} nx^{n-1} \right) = -\frac{1}{2} \sum_{n=2}^{\infty} (-1)^{n+1} n(n-1)x^{n-2} = \sum_{n=2}^{\infty} (-1)^{n+2} \frac{n}{2} (n-1)x^{n-2}.\]

(c) The power series of
\[f(x) = \frac{x^2}{(1+x)^3} = x^2 \left( \frac{1}{(1+x)^3} \right)\]
is (using (b))
\[x^2 \sum_{n=2}^{\infty} (-1)^{n+2} \frac{n}{2} (n-1)x^{n-2} = \sum_{n=2}^{\infty} (-1)^{n+2} \frac{n}{2} (n-1)x^n.\]

16. Find the power series representation and the radius of convergence of the function
\[f(x) = \frac{x^2}{(1-2x)^2}.\]

**Solution:** This is similar to Problem 13. We notice that
\[f(x) = x^2 \left( \frac{1}{(1-2x)^2} \right) = (x^2) \left( \frac{1}{2} \right) \frac{d}{dx} \left( \frac{1}{1-2x} \right)\]
and we know that the power series representation of \(\frac{1}{1-2x}\) is
\[\sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n\]
which has radius of convergence \(R = \frac{1}{2}\). Thus, the power series representation of \(f(x)\) is
\[\frac{x^2}{2} \frac{d}{dx} \left( \sum_{n=0}^{\infty} 2^n x^n \right) = \frac{x^2}{2} \sum_{n=1}^{\infty} 2^n nx^{n-1} = \sum_{n=1}^{\infty} 2^{n-1} nx^n.\]
The radius of convergence is \(R = \frac{1}{2}\).

23. Evaluate the following indefinite integral as a power series.
\[\int \frac{t}{1-t^8} dt.\]

**Solution:** The integrand is \(t \left( \frac{1}{1-t^8} \right)\). The power series representation of \(\frac{1}{1-t^8}\) is
\[\sum (t^8)^n = \sum t^{8n}\]
Thus,
\[ \int \frac{t}{1 - t^8} dt = \int \sum t^{8n+1} dt = C + \sum \frac{t^{8n+2}}{8n+2} \]
where \( C \) is a constant of integration.

**Section 12.10**

Find the Maclaurin Series of each of the following functions by using the definition of the Maclaurin Series.

5. \( f(x) = (1 + x)^{-3} \).

**Solution:** By definition, the Maclaurin series of a function \( f \) is
\[ \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n. \]
We calculate
\[ f'(x) = -3(1 + x)^{-4} \]
\[ f^{(2)}(x) = (-3)(-4)(1 + x)^{-5} = 12(1 + x)^{-5} \]
\[ f^{(3)}(x) = (12)(-5)(1 + x)^{-6} \]
\[ \vdots \]
\[ f^{(n)}(x) = (-1)^n \frac{(n + 2)!}{2} (1 + x)^{-3-n}. \]
Thus, we have
\[ \frac{f^{(n)}(0)}{n!} = (-1)^n \frac{(n + 2)!}{2(2n)!} = (-1)^n (n + 2)(n + 1). \]
The Maclaurin series of \( f \) is
\[ \sum_{n=0}^{\infty} (-1)^n \frac{(n + 2)(n + 1)}{2} x^n. \]

Using Ratio test, we can find that \( R = 1 \).

8. \( f(x) = xe^x \).
Solution: Once again we calculate the derivatives
\[ f'(x) = e^x + xe^x = (1 + x)e^x \]
\[ f''(x) = e^x + (1 + x)e^x = (2 + x)e^x \]
\[ f'''(x) = e^x + (2 + x)e^x = (3 + x)e^x \]
\[ \vdots \]
\[ f^{(n)}(x) = (n + x)e^x. \]
Thus
\[ \frac{f^{(n)}(0)}{n!} = \frac{n}{n!} = \frac{1}{(n-1)!} \]
and the Maclaurin series is
\[ \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}. \]
We can also write the series as
\[ \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}. \]
This is the series you get if you simply multiply \( x \) and the Maclaurin series of \( e^x \).

In the following, find the Taylor series expansion of the given function about a given \( a \).

11. \( f(x) = 1 + x + x^2, \quad a = 2. \)

Solution: We can’t use any of our tricks here so we must find the Taylor series using the actual definition. The Taylor series of \( f \) about \( a = 2 \) is
\[ \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x - 2)^n. \]
Let’s calculate the various derivatives
\[ f'(x) = 1 + 2x \]
\[ f''(x) = 2 \]
\[ f^{(n)}(x) = 0, \quad n > 2. \]
This gives us \( f(2) = 1 + 2 + 2^2 = 7, \quad f'(2) = 5, \quad f''(2) = 2 \) and \( f^{(n)}(2) = 0 \) for \( n > 2. \)
Thus, the Taylor series of \( f \) about \( a = 2 \) is
\[ f(2) + f'(2)(x - 2) + \frac{f''(2)}{2!} (x - 2)^2 = 7 + 5(x - 2) + (x - 2)^2. \]
17. \( f(x) = \frac{1}{\sqrt{x}} \), \( a = 9 \).

**Solution:** Here again we can’t use any tricks so let’s calculate the derivatives.

\[
\begin{align*}
    f'(x) &= -\frac{1}{2}x^{-3/2} \\
    f''(x) &= \left(-\frac{1}{2}\right)\left(-\frac{3}{2}x^{-5/2}\right) = \frac{3}{4}x^{-5/2} \\
    f'''(x) &= \left(\frac{3}{4}\right)\left(-\frac{5}{2}x^{-7/2}\right) = -\frac{15}{8}x^{-7/2}.
\end{align*}
\]

Since it is not obvious what the general pattern for \( f^{(n)} \) is we shall just write the first 3 terms of the Taylor series. We have \( f(9) = \frac{1}{3} \), \( f'(9) = -\frac{1}{2(3^{3/2})} f''(9) = \frac{3}{4(3^{5/2})} \) and so on. Thus the Taylor is

\[
f(9) + f'(9)(x-9) + \frac{f''(9)}{2!}(x-9)^2 + \ldots = \frac{1}{3} - \frac{1}{2(3^{3/2})}(x-9) + \frac{1}{8(3^{4})}(x-9)^2 + \ldots
\]

28. Use the Maclaurin Series derived in class to obtain the Maclaurin Series of \( f(x) = x \cos 2x \).

**Solution:** We have

\[
x \cos 2x = x \sum_{n=0}^{\infty} \frac{(2x)^{2n}}{(2n)!} = x \sum_{n=0}^{\infty} \frac{2^{2n}x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{2^{2n}x^{2n+1}}{(2n)!}.
\]

41. Evaluate the following indefinite integral as a power series.

\[
\int \sqrt{1 + x^3} \, dx.
\]

**Solution:** We find the power series of the function \( f(x) = \sqrt{1 + x^3} \). Since none of our tricks will work, we find the Maclaurin series using the definition. We calculate

\[
\begin{align*}
    f'(x) &= \frac{3}{2}x^2(1 + x^3)^{-1/2} \\
    f''(x) &= 3x(1 + x^3)^{-1/2} - \frac{3x^2}{2}(1 + x^3)^{-3/2} \\
    f'''(x) &= 3(1 + x^3)^{-1/2}
\end{align*}
\]

We won’t calculate these derivatives any further. The Maclaurin series of \( f \) is

\[
1 + \frac{3}{3!}x^3 + \ldots = 1 + \frac{x^3}{2} + \ldots
\]
Hence

\[ \int \sqrt{1 + x^3} \, dx = \int (1 + \frac{x^3}{2} + ...) \, dx = C + x + \frac{x^4}{8} + ... \]