Math 2433 Homework #3

Solutions

Section 12.3.

Use the integral test to determine whether the series is convergent or divergent.

8. $\sum \frac{n+2}{n+1}$.

**Solution:** To use the integral test, we must first verify that the associated function $f(x) = \frac{x+2}{x+1}$ is continuous, positive and decreasing. The function is clearly continuous and positive on $[1, \infty)$. To check if it is decreasing, we calculate

$$f'(x) = -\frac{1}{(x+1)^2}.$$  

Since $f'(x) < 0$ for all $x$, the function $f$ is decreasing. Next, we calculate

$$\int_1^t \frac{x+2}{x+1} \, dx = \int_1^t \left(1 + \frac{1}{x+1}\right) \, dx = [x + \ln(x+1)]_1^t = (t - 1) + \ln \left(\frac{t+1}{2}\right).$$

The improper integral is

$$\int_1^\infty = \lim_{t \to \infty} \left(t - 1 + \ln \frac{t+1}{2}\right) = \infty$$

so it diverges. By the Integral Test the series $\sum \frac{n+2}{n+1}$ also diverges.

In the following, determine whether the series is convergent or divergent.

13. $\sum \frac{5 - 2\sqrt{n}}{n^3}$.

**Solution** Notice that the associated function

$$f(x) = \frac{5 - 2\sqrt{x}}{x^3}$$

is not positive on all of $[1, \infty)$ so we cannot use the Integral Test here.

We have

$$\frac{5 - 2\sqrt{n}}{n^3} = \frac{5}{n^3} - \frac{2\sqrt{n}}{n^3} = \frac{5}{n^3} - \frac{2}{n^{5/2}}.$$  

Now both $\sum \frac{5}{n^3}$ and $\sum \frac{2}{n^{5/2}}$ are convergent series (each is a $p$-series with $p > 1$). Therefore the given series $\sum \frac{5 - 2\sqrt{n}}{n^3}$ is convergent since it is the difference of two convergent series.
Note that we could also have used the limit comparison test here. Since the dominant terms in the numerator and denominator are $-2\sqrt{n}$ and $n^3$ respectively, we can use the series $\sum \frac{-2\sqrt{n}}{n^3} = -\sum \frac{2}{n^{3/2}}$, which we know converges. We calculate

$$\lim_{n \to \infty} \left( \frac{\frac{5}{2\sqrt{n}}}{\frac{5}{\sqrt{n}}} \right) = \lim_{n \to \infty} \frac{2\sqrt{n} - 5}{2\sqrt{n}} = \lim_{n \to \infty} \left( 1 - \frac{5}{2\sqrt{n}} \right) = 1.$$

Since the limit exists and is positive, by Limit Comparison Test, the given series is convergent.

16. $\sum \frac{3n+2}{n(n+1)}$

**Solution:** Since the limit

$$\lim_{n \to \infty} \frac{3n + 2}{n(n+1)} = \lim_{n \to \infty} \frac{\frac{3}{n} + \frac{2}{n^2}}{1 + \frac{1}{n}} = 0$$

we cannot use the Test for Divergence. The term $\frac{3n+2}{n(n+1)}$ can be decomposed as a sum of partial fractions as follows.

$$\frac{3n+2}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

where $A$ and $B$ are constants to be determined. Taking the common denominator on the right-hand side, we get

$$\frac{3n+2}{n(n+1)} = \frac{A(n+1) + Bn}{n(n+1)} = \frac{(A+B)n + A}{n(n+1)}.$$

Thus, we must have $A = 2$ and $A + B = 3$, so $B = 1$. Thus

$$\frac{3n+2}{n(n+1)} = \frac{2}{n} + \frac{1}{n+1}.$$

Now, the associated function is

$$f(x) = \frac{2}{x} + \frac{1}{x+1}$$

which is clearly continuous and positive on $[1, \infty)$. Also,

$$f'(x) = -\frac{2}{x^2} - \frac{1}{(1+x)^2}$$

so $f'(x) < 0$ and $f$ is decreasing. We can now apply the Integral Test. The improper integral is

$$\int_{1}^{\infty} f(x)dx = \lim_{t \to \infty} \int_{1}^{t} \left( \frac{2}{x} + \frac{1}{x+1} \right)dx = \lim_{t \to \infty} [2 \ln x + \ln(1+x)]_{1}^{t} = \lim_{t \to \infty} \left( \ln \frac{t^2(1+t)}{2} \right),$$
which diverges. Hence by the Integral Test, the given series diverges.

Again, we could also have used the Limit Comparison Test here. Consider the series \( \sum \frac{3n+2}{n^2+n} \). The dominant terms in the numerator and denominator are \( 3n \) and \( n^2 \) respectively, so we can use the series \( \sum \frac{3n}{n^2} = \sum \frac{3}{n} \) (which we know diverges) to apply the limit comparison test. We have

\[
\lim_{n \to \infty} \frac{\frac{3n+2}{n^2+n}}{\frac{3n}{n^2}} = \lim_{n \to \infty} \frac{\frac{3n+2}{3n}}{\frac{n^2+n}{n^2}} = \lim_{n \to \infty} \frac{1 + \frac{2}{3n}}{1 + \frac{1}{n}} = 1.
\]

Since the limit exists and is positive, the given series diverges by the Limit Comparison Test.

In the following, find the value of \( p \) for which the series is convergent.

25. \( \sum \frac{1}{n(ln n)^p} \)

**Solution:** The associated function is

\[ f(x) = \frac{1}{x(ln x)^p}. \]

This function is clearly continuous and positive on \((1, \infty)\). We calculate

\[ f'(x) = -\frac{(ln x)^p + p(ln x)^{p-1}}{(x(ln x)^p)^2} \]

so \( f'(x) < 0 \) and the function is decreasing. We can now apply the Integral Test. The improper integral is

\[ \int_1^\infty \frac{1}{x(ln x)^p} \, dx = \lim_{t \to \infty} \int_1^t \frac{1}{x(ln x)^p} \, dx. \]

We make the substitution \( u = \ln x \). Then \( dx = x \, du \). We thus have

\[
\int_1^t \frac{1}{x(ln x)^p} \, dx = \int_0^{\ln t} \frac{1}{(xu)^p} \, xdu = \int_0^{\ln t} u^{-p} \, du = \left[ \frac{u^{1-p}}{-p+1} \right]_0^{\ln t} = \frac{(\ln t)^{1-p}}{-p+1} = \frac{1}{1-p} \left( \frac{1}{(\ln t)^{p-1}} \right).
\]

Thus,

\[ \int_1^\infty \frac{1}{x(ln x)^p} \, dx = \frac{1}{1-p} \left( \lim_{t \to \infty} \frac{1}{(\ln t)^{p-1}} \right) \]

which converges for \( p > 1 \). Thus, the given series is convergent for \( p > 1 \).
30. (a) Find the partial sum $s_{10}$ of the series $\sum \frac{1}{n^4}$. Estimate the error in using $s_{10}$ as an approximation to the sum of the series.
(b) Improve the estimate with (3) with $n = 10$.
(c) Find the value of $n$ so that $s_n$ is within 0.00001 of the sum.

Solution: (a) The series is a $p$-series with $p = 4$, so it converges. Using a calculator we determine that $s_{10} = 1.082$. Now, the function associated with the given series is $f(x) = \frac{1}{x^4}$. This is clearly continuous and positive on $[1, \infty)$. Also, 
$$f'(x) = -4x^{-5} < 0 \text{ on } [1, \infty)$$
so $f$ is decreasing. If the given series converges to $s$, the error (or the remainder) $R_{10}$ is given by $R_{10} = s - s_{10}$.

We calculate 
$$\int_{n}^{\infty} \frac{1}{x^4} \, dx = \lim_{t \to \infty} \int_{n}^{t} \frac{1}{x^4} \, dx = \lim_{t \to \infty} \left[ -\frac{1}{3x^3} \right]_{n}^{t} = \frac{1}{3n^3}. $$

By the Theorem on estimates of sum of a series, we know that 
$$\int_{n+1}^{\infty} \frac{1}{x^4} \, dx \leq R_{n} \leq \int_{n}^{\infty} \frac{1}{x^4} \, dx.$$

In other words, 
$$\frac{1}{3(10 + 1)^3} \leq R_{10} \leq \frac{1}{3(10^5)}.$$

The error is thus 
$$0.00025 \leq R_{10} \leq 0.00033.$$

(b) Using (3) in the book, we have 
$$s_{10} + \frac{3}{11^3} \leq s \leq s_{10} + \frac{3}{10^3}$$

substituting $s_{10} = 1.082$ we get 
$$1.08225 \leq s \leq 1.08233.$$

(c) We want to find $n$ such that 
$$\frac{1}{3n^3} \leq 0.00001.$$ 

That is, 
$$n^3 \geq \frac{10^5}{3}$$

which gives $n \geq 32.18$. In other words, $n$ has to be at least 33 to achieve the desired accuracy.