Math 2433 Homework #6

Solutions

Important: I will only provide helpful hints and not detailed solutions, but you have to write all the steps involved along with careful explanations.

Section 12.8.
(We denote the radius of convergence by \( R \) throughout this document.)
Find the radius of convergence and interval of convergence of the series.
8. \( \sum n^n x^n \).

Solution:
We find that \( \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} n|x| = \infty, \ x \neq 0 \). Thus, by Root Test the given series diverges for all \( x \neq 0 \). At \( x = 0 \) it converges since the series is centered at this point. Thus \( R = 0 \) and interval of convergence is \( \{0\} \).

16. \( \sum (-1)^n \frac{(x-3)^n}{2n+1} \).

Solution: We have \( a_n = (-1)^n \frac{(x-3)^n}{2n+1} \) so that
\[
\frac{|a_{n+1}|}{a_n} = |x-3| \left( \frac{2n+1}{2n+3} \right)
\]
and thus
\[
\lim_{n \to \infty} \frac{|a_{n+1}|}{a_n} = |x-3|.
\]
By Ratio Test the series converges if \( |x-3| < 1 \). Thus, the radius of convergence is \( R = 1 \) and the interval of convergence contains \((2, 4)\). We now determine what happens at the end points.
For \( x = 2 \) the given series becomes \( \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = \sum_{n=0}^{\infty} \frac{1}{2n+1} \), which is divergent by Limit Comparison Test (use \( b_n = \frac{1}{2n} \)). I am not writing the details but you have to do so. For \( x = 4 \) the series becomes \( \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \) which can be shown to be convergent by the Alternating Series Test. Hence, the interval of convergence is \((2, 4]\).

18. \( \sum_{n=1}^{\infty} \frac{n}{2}(x+1)^n \).

Solution: Here \( a_n = \frac{n}{2}(x+1)^n \) and thus
\[
\lim_{n \to \infty} \frac{|a_{n+1}|}{a_n} = \lim_{n \to \infty} \left( \frac{|x+1|}{n+1} \right) = |x+1|
\]
thus by Ratio Test, the series converges for \( |x+1| < 1 \) so \( R = 1 \) and the interval contains \((-2, 0]\).
At \( x = -2 \) the series is \( \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{2} \right) \) which is divergent by Test for Divergence.
At \( x = 0 \) the series is \( \sum_{n=1}^{\infty} \frac{n}{4} \) which also diverges by Test for Divergence. Thus, the interval of convergence is \((-2, 0)\).

20. \( \sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n} \).

**Solution:** Here \( a_n = \frac{(3x-2)^n}{n3^n} \) thus

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{3|x - \frac{2}{3}|}{n+1} = \left| x - \frac{2}{3} \right|
\]

and by Ratio Test the series converges for \( \left| x - \frac{2}{3} \right| < 1 \) Hence, \( R = 1 \) and the interval of convergence contains \((-\frac{1}{3}, \frac{5}{3})\).

Next, we check what happens at the end points. At \( x = -\frac{1}{3} \), the series is \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) which is convergent by Alternating Series Test. At \( x = \frac{5}{3} \) the series is \( \sum_{n=1}^{\infty} \frac{1}{n} \) which is divergent. Hence the interval of convergence is \([\frac{1}{3}, \frac{5}{3})\).

22. \( \sum_{n=1}^{\infty} \frac{n(x-4)^n}{n^3+1} \).

**Solution:** We have \( a_n = \frac{n(x-4)^n}{n^3+1} \) and thus

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left( \frac{n}{n+1} \right) \left( \frac{n^3+1}{(n+1)^3+1} \right) \left| x - 4 \right| = \left( \frac{1}{1 + \frac{1}{n}} \right) \left( \frac{1 + \frac{n}{1+n^3}}{1 + \frac{n}{n^3}} \right) \left| x - 4 \right|.
\]

Thus,

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| x - 4 \right|.
\]

By Ratio Test the series converges for \( \left| x - 4 \right| < 1 \) and thus \( R = 1 \) and the interval of convergence contains \((3, 5)\).

You can check that the series converges at \( x = 3 \) by Alternating Series Test and also at \( x = 5 \) by Limit Comparison Test (you have to fill in the details) so the interval of convergence is \([3, 5]\).

24. \( \sum_{n=1}^{\infty} \frac{n^2 x^n}{2 \cdot 4 \cdot 6 \cdots (2n)} \).

**Solution:** Notice that

\[
a_n = \frac{n^2 x^n}{2 \cdot 4 \cdot 6 \cdots (2n)} = \frac{n^2 x^n}{2^n n!}.
\]

Therefore

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left( \frac{n+1}{n} \right)^2 \frac{1}{n+1} \frac{|x|}{2} = 0.
\]

and by Ratio Test the series converges for all \( x \). The radius of convergence is \( R = \infty \) and the interval of convergence is \((-\infty, \infty)\).
Section 12.9
Find a power series representation for the function and determine the radius of convergence.

4. \( f(x) = \frac{3}{1-x^4} \).
Solution: We have \( f(x) = 3 \frac{1}{1-x^4} \) and thus a power series representation of \( f(x) \) is
\[
3 \sum_{n=0}^{\infty} (x^4)^n = \sum_{n=0}^{\infty} 3x^{4n}.
\]
The series converges for \(|x^4| < 1\), or \(|x| < 1\). Thus, \( R = 1 \).

8. \( f(x) = \frac{x}{2x^2+1} \).
Solution: The function is \( f(x) = x \frac{1}{1+2x^2} \) and a power series representation is
\[
x \sum_{n=0}^{\infty} (-2x^2)^n = \sum_{n=0}^{\infty} (-1)^n 2^n x^{2n+1}.
\]
The series is convergent for \(|-2x^2| < 1\) or \(|x| < \frac{1}{\sqrt{2}}\), so \( R = \frac{1}{\sqrt{2}} \).

14. (a) Find a power series representation for \( f(x) = \ln(1 + x) \). What is the radius of convergence?
(b) Use part (a) to find a power series for \( f(x) = x \ln(1 + x) \).
(c) Use part (a) to find a power series for \( f(x) = \ln(x^2 + 1) \).
Solution: (a) We have \( f(x) = \int \frac{1}{1+x} \) and the integrand can be expressed as a power series. Thus, a power series for \( f(x) \) is
\[
\int \frac{1}{1+x} \, dx = \int \sum_{n=0}^{\infty} (-x)^n \, dx = \sum_{n=0}^{\infty} (-1)^n x^{n+1} + C.
\]
At \( x = 0 \) the left-hand side is zero and the right-hand side is \( C \), so \( C = 0 \). The radius of convergence of the series remained unchanged after integration, so \( R = 1 \).

(b) A power series for \( f(x) \) is \( x \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n x^{n+2} \).

(c) This is straightforward. A power series for \( f(x) \) is
\[
\sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n x^{2n+2} \frac{x^{n+2}}{n+1}.
\]
16. Find a power series representation for the function and determine the radius of convergence. \( f(x) = \frac{x^2}{(1-2x)^2} \).
Solutions

Solution: Note that \( f(x) = x^2 \left( \frac{1}{(1-2x)^2} \right) \) so the main part of the problem is to find a power series for \( \frac{1}{(1-2x)^2} \). Observe that

\[
f(x) = x^2 \frac{1}{(1-2x)^2} = \left( x^2 \right) \frac{1}{2} \frac{d}{dx} \left( \frac{1}{1-2x} \right)
\]

and a power series for \( \frac{1}{1-2x} \) is \( \sum_{n=0}^{\infty} 2^n x^n \). Hence, a power series for \( f(x) \) is

\[
\frac{x^2}{2} \sum_{n=1}^{\infty} 2^n n x^{n-1} = \sum_{n=1}^{\infty} 2^{n-1} n x^{n+1}.
\]

The radius of convergence is \( R = \frac{1}{2} \).

Evaluate the indefinite integral as a power series. What is the radius of convergence?

24. \( \int \ln(1-t) dt \).

Solution: We have, from previous work in class, that a power series for \( \ln(1-t) \) is \( -\sum_{n=0}^{\infty} \frac{t^n}{n+1} \) and hence

\[
\int \frac{\ln(1-t)}{t} dt = \int -\sum_{n=0}^{\infty} \frac{t^n}{n+1} dt = -\sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)^2} + C
\]

The radius of convergence is the same as that of the original series, so \( R = 1 \).

26. \( \int \tan^{-1}(x^2) dx \).

I will let you answer this. Substitute \( x^2 \) for \( x \) in the series for \( \tan^{-1} x \) and integrate.