Math 2433 Homework #4

Solutions

Important: I will only provide helpful hints and not detailed solutions, but you have to write all the steps involved along with careful explanations.

Section 12.4.

Determine whether the series converges or diverges.

6. $\sum \frac{n-1}{n^{\sqrt{n}}}$

Solution: Use comparison test with $\sum b_n = \sum \frac{n}{n^{\sqrt{n}}}$ which is a convergent $p$ series. Thus the given series converges.

10. $\sum \frac{n^2 - 1}{3n^3 + 1}$

Solution: Consider the series $\sum b_n = \sum \frac{n^2}{3n^3} = \sum \frac{1}{3n}$ which is convergent. We have

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^2 - 1}{3n^3} \cdot \frac{3n^4 + 1}{1} = 1 > 0$$

so by Limit Comparison Test, the given series converges.

12. $\sum \frac{1 + \sin n}{10^n}$

Solution: We have

$$\frac{1 + \sin n}{10^n} < \frac{2}{10^n}$$

and $\sum \frac{2}{10^n}$ is convergent, so by Comparison Test, the given series converges.

16. $\sum \frac{1}{\sqrt{n^3} + 1}$

Solution: Use Comparison Test with $\sum b_n = \frac{1}{\sqrt{n^3}} = \sum \frac{1}{n^{3/2}}$. We have

$$\frac{1}{\sqrt{n^3} + 1} < \frac{1}{\sqrt{n^3}}$$

which means that the series converges.

20. $\sum_{n=1}^{\infty} \frac{n + 4^n}{n^6}$

Solution: Consider the series $\sum b_n = \sum \frac{4^n}{6^n}$, which is a convergent geometric series. We calculate

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n + 4^n}{n^6} \cdot \frac{6^n}{4^n} = \lim_{n \to \infty} \frac{1 + \frac{n}{4^n}}{1 + \frac{n}{6^n}} = 1$$
where we have used L'Hospital Rule to conclude that \( \lim_{n \to \infty} \frac{n}{\sqrt[n]{n^4}} = 0 = \lim_{n \to \infty} \frac{n}{\sqrt{n^6}}. \)

Thus, by Limit Comparison Test, the given series converges.

26. \( \sum_{n=1}^{\infty} \frac{n+5}{\sqrt[n]{n^4} + n^2} \).

**Solution:** This is straightforward. Look at the dominant terms in the numerator and
denominator and consider \( \sum b_n = \sum \frac{n}{\sqrt[n]{n^4}} = \sum \frac{1}{n^4}, \) which is convergent. Now

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n + 5}{n} \left( \frac{\sqrt[n]{n^4}}{\sqrt[n]{n^4} + n^2} \right) = 1
\]

so by Limit Comparison Test, the given series converges.

12.5

Test the series for convergence or divergence.

4. \( \frac{1}{\sqrt[4]{2}} - \frac{1}{\sqrt[4]{3}} + \frac{1}{\sqrt[4]{4}} - \frac{1}{\sqrt[4]{5}} + \frac{1}{\sqrt[4]{6}}. \)

**Solution:** The series is \( \sum_{n=2}^{\infty} (-1)^n \frac{1}{\sqrt[4]{n}}. \) This is an alternating series of the form
\( \sum (-1)^n b_n \) where \( b_n = \frac{1}{\sqrt[4]{n}}. \) Now, clearly \( \lim_{n \to \infty} b_n = 0. \) Also, \( \frac{1}{\sqrt[4]{n+1}} < \frac{1}{\sqrt[n]{n}}. \) Thus, by
Alternating Series Test, the given series converges.

6. \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)}. \)

**Solution:** This is of the form \( \sum (-1)^n b_n \) where \( b_n = \frac{1}{\ln(n+4)}. \) Again, it is clear that
\( \lim_{n \to \infty} b_n = 0. \) Now, to show that \( b_n \) is decreasing we look at the function \( f(x) = \frac{1}{\ln(x+4)}. \)
We have

\[
f'(x) = \frac{-1}{(\ln(x+4))^2} < 0
\]

so \( f \) is decreasing and so is \( b_n. \) By Alternating Series Test, the given series converges.

8. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n \sqrt[n]{n^2 + 2}}. \)

**Solution:** Let \( b_n = \frac{n}{\sqrt[n]{n^2 + 2}}. \) Then, \( \lim_{n \to \infty} \frac{n}{\sqrt[n]{n^2 + 2}} = \lim_{n \to \infty} \frac{n}{\sqrt[n]{1 + \frac{2}{n^2}}} = 0. \) Next, to show
that \( b_n \) is decreasing, we can find the derivative of \( f(x) = \frac{\sqrt[4]{x}}{\sqrt{x^2 + 2}} \) and show that it is
negative.

10. \( \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1 + 2 \sqrt{n}}. \)

**Solution:** Let \( b_n = \frac{\sqrt{n}}{1 + 2 \sqrt{n}}. \) Again, we can easily show that \( \lim_{n \to \infty} b_n = 0. \) However, for function
\( f(x) = \frac{\sqrt{x}}{1 + 2 \sqrt{x}} \) we have

\[
f'(x) = \frac{1}{(1 + 2 \sqrt{x})^2} > 0.
\]
so $f$ is increasing and $b_n$ is increasing. This means that we cannot use the Alternating Series Test. However, we see that $\lim_{n \to \infty} (-1)^n \frac{\sqrt{n}}{1 + 2\sqrt{n}}$ does not exist. Hence, the series diverges by the Test for divergence.

12. $\sum (-1)^{n-1} \frac{e^{1/n}}{n}$
Solution: Let $b_n = \frac{e^{1/n}}{n}$. It is easy to see that $\lim_{n \to \infty} b_n = 0$. Also, for $f(x) = \frac{e^{1/x}}{x}$, we have
$$f'(x) = \frac{\left(-\frac{1}{x^2}e^{1/x}\right)x - e^{1/x}}{x^2} = -\frac{(1 + x)e^{1/x}}{x^3} < 0$$
so $b_n$ is decreasing, and by Alternating Series Test, the given series converges.

14. $\sum (-1)^{n-1} \frac{\ln n}{n}$
Solution: This is very similar to a problem given in Midterm 2. The series is convergent by Alternating Series Test. The key is to find the derivative of $f(x) = \frac{\ln x}{x}$. We have
$$f'(x) = \frac{1 - \ln x}{x^2} < 0, \quad x \geq 3$$
so $b_n$ is decreasing. It is easy to see that $\lim_{n \to \infty} b_n = 0$ by using L’Hospital Rule.