Section 12.1.
since \( \lim_{n \to \infty} \frac{1}{n} = 0 \).

Determine whether the following sequence converges or diverges. If it converges, find the limit. \( a_n = \frac{n!}{2^n} \).

Solution: Let’s first look at the first few terms of the sequence:

\[
\frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{2}, \frac{15}{4}, \ldots
\]

It certainly appears that the terms in the sequence get bigger and bigger as \( n \) increases. More precisely, we have

\[
\frac{n!}{2^n} = \frac{n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1}{2 \cdot 2 \cdot 2 \cdots 2 \cdot 2} = \left(\frac{n}{2}\right) \cdot \left(\frac{n-1}{2}\right) \cdot \left(\frac{n-2}{2}\right) \cdots \left(\frac{2}{2}\right) \cdot \left(\frac{1}{2}\right)
\]

(0.1)

Now, except for the last two terms, each term in the product on the right-hand side is greater than 1. That is,

\[
\left(\frac{n-j}{2}\right) > 1, \quad n > 2, \quad 0 \leq j < n.
\]

Thus the product on the right-hand side of (0.1) tends to infinity as \( n \to \infty \) and the sequence diverges.

Determine whether the sequence is increasing, decreasing or not monotonic. Is the sequence bounded?

60. \( \{(-2)^{n+1}\} \).

Solution: \( a_n = (-2)^{n+1} \) so

\[
a_{n+1} - a_n = (-2)^{n+2} - (-2)^{n+1} = (-2)^n(-2)^2 - (-2)^n(-2) = 6(-2)^n
\]

so \( a_{n+1} - a_n > 0 \) if \( n \) is even and \( a_{n+1} - a_n < 0 \) if \( n \) is odd. So the sequence is neither increasing nor decreasing so it is not monotonic. The sequence is not bounded as it is not bounded above.
62. \( a_n = \frac{2n-3}{3n+4} \).

Solution: We have

\[
a_{n+1} - a_n = \frac{2(n+1) - 3}{3(n+1) + 4} - \frac{2n - 3}{3n + 4} = \frac{(2n - 1)(3n + 4) - (2n - 3)(3n + 7)}{(3n + 4)(3n + 7)} = \frac{-17}{(3n + 4)(3n + 7)} < 0, \quad n \geq 1.
\]

and thus the sequence is decreasing. Also, \( \lim_{n \to \infty} a_n = \frac{2}{3} \) so the sequence is bounded above. Also, clearly \( a_n > 0 \) for \( n \geq 1 \). Thus the sequence is also bounded below and hence it is bounded.

64. \( a_n = ne^{-n} \).

Solution: We look at the corresponding function \( f(x) = xe^{-x} \). The derivative is

\[
f'(x) = e^{-x} - xe^{-x} = (1 - x)e^{-x}
\]

so \( f'(x) < 0 \) for \( x > 1 \) and the function is decreasing on \((1, \infty)\). Consequently, the given sequence is also decreasing. Since the sequence is decreasing, it is bounded above by \( \frac{1}{e} \) and it is bounded below by 0, so it is bounded.

\[a_n = \frac{n}{n^2 + 1} \]

Solution: Method 1: We have

\[
a_{n+1} - a_n = \frac{n + 1}{(n + 1)^2 + 1} - \frac{n}{n^2 + 1} = \frac{n + 1}{n^2 + 2n + 2} - \frac{n}{n^2 + 1} = \frac{n^3 + n^2 + n + 1 - n^3 - 2n^2 - 2n}{(n^2 + 2n + 2)(n^2 + 1)} = \frac{-1}{(n^2 + 2n + 2)(n^2 + 1)} < 0
\]

since \( 1 < (n + n^2) \) for all \( n \). Hence \( (a_{n+1} - a_n) < 0 \) and the sequence is decreasing.

Method 2: Define a function \( f \) by

\[f(x) = \frac{x}{x^2 + 1} \]

We calculate

\[
f'(x) = \frac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}
\]
Thus, $f'(x) < 0$ for $x > 1$ and the function $f$ is decreasing on $(1, \infty)$. We must therefore have that $a_n = f(n) = \frac{n}{n^2 + 1}$ is decreasing.

Next, it is clear that, for $n \geq 1$ we have $\frac{n}{n^2 + 1} > 0$ so $a_n$ is bounded below by 0. Also, we have $n^2 + 1 > n$ for all $n \geq 1$, so $a_n$ is bounded above by 1.

66. $n + \frac{1}{n}$.

**Solution:** This should be obvious now. The sequence is increasing but not bounded (as it is not bounded above).