1. Let \((X, B, \mu)\) be a measure space, and \(<E_i>\) be a sequence of sets in \(B\). Prove that
\[
\mu(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu E_i
\]

2a. State Hahn-Banach Theorem.

2b. Let \(X\) be a normed vector space and \(X^{**}\) be the dual of the dual of \(X\). Prove by an application of Hahn-Banach theorem that there is a natural isomorphism from \(X\) into a linear subspace of \(X^{**}\).

3a. Prove that if \(<f_n>\) is an equicontinuous sequence of mapping from a metric space \(X\) to a complete metric space \(Y\), and if the sequences \(<f_n(x)>\) converge for each \(x\) of a dense subset \(D\) of \(X\), then \(<f_n>\) converges at each point of \(X\), and the limit function is continuous.

3b. State Ascoli-Arzelá Theorem.
Show that the Ascoli-Arzelá Theorem may fail if the various hypotheses are dropped for the following sequences of functions:

3c. \(f_n(x) = x + n\) for \(x \in [0, 1]\).

3d. \(f_n(x) = x^n\) for \(x \in [0, 1]\).

4. State and Prove Riesz representation theorem in the classical Banach spaces: Let \(F\) be a bounded linear functional on \(L^p, 1 \leq p < \infty\). Then there exists a unique function \(g\) in \(L^q\) such that
\[
F(f) = \int fg.
\]
We also have $||F|| = ||g||_q$.

5. Prove the Projection Theorem in a Hilbert space:
Let $S$ be a closed subspace of a Hilbert space $H$, and $S^\perp$ be the set of elements orthogonal to every element of $S$. Then for every $x \in H$ we have $x = y + z$ where $y \in S$ and $z \in S^\perp$.

6a. State Fubini’s Theorem

6b. Let

$$f(x, y) = \begin{cases} 
  y^2 & \text{if } 0 < x < y < 1, \\
  -x^2 & \text{if } 0 < y < x < 1, \\
  0 & \text{otherwise if } 0 \leq x \leq 1, 0 \leq y \leq 1.
\end{cases}$$

Compute

$$\int_0^1 \int_0^1 f(x, y) \, dy \, dx$$

and

$$\int_0^1 \int_0^1 f(x, y) \, dx \, dy.$$

(Hint $\int_0^1 f(x, y) \, dx = \int_0^y \frac{dy \, dx}{y^2} = \int_0^1 f(x, y) \, dy$)

6c. Is it true that $\int_0^1 \int_0^1 f(x, y) \, dy \, dx = \int_0^1 \int_0^1 f(x, y) \, dx \, dy$? Is Fubini’s Theorem applicable?

Let $<X, p>$ be a metric space.
Prove that the following four statements are equivalent:

7a. $<X, p>$ is compact.

7b. $<X, p>$ is sequentially compact.

7c. $<X, p>$ has the Bolzano-Weierstrass property.

7d. $<X, p>$ is complete and totally bounded.

8a. Prove Young’s inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

where $a, b \geq 0, 1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1$. 

8b. Use 8a, Young’s inequality to give a proof of the Hölder inequality.

9a. If $f(x)$ is a function of bounded variation, then is it necessary that $f(x)$ has to be bounded?

9b. Let $f(x) = x \cos(\pi/2x)$ if $x \neq 0$, $f(0) = 0$. Prove or disprove that $f$ is of bounded variation in $[0,1]$.

10. Prove the following generalization of Lebesgue Convergence Theorem: Let $\langle g_n \rangle$ be a sequences of integrable functions which converges a.e. to an integrable function $g$. Let $\langle f_n \rangle$ be a sequences of measurable functions such that $|f_n| \leq g_n$, and $\langle f_n \rangle$ converges to $f$ a.e. If

$$\int g = \lim \int g_n.$$ 

then

$$\int f = \lim \int f_n.$$