1a. Prove that 
\[ m(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} mE_i \]

1b. If the sets \( E_n \) are pairwise disjoint, prove that 
\[ m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} mE_i \]

2a. State Hahn-Banach Theorem.

2b. Use the Hahn-Banach Theorem to prove that the Riesz Representation Theorem does not hold for bounded linear functional on \( L^{\infty}[0, 1] \), or the dual space of \( L^{\infty}[0, 1] \) is not \( L^{1}[0, 1] \).

3a. Prove that if \(<f_n>\) is a sequence of mappings of a countable set \( D \) into a metric space \( Y \) such that for each \( x \in D \) the closure of the set \( \{f_n(x) : 0 \leq n < \infty\} \) is compact, then there is a subsequence that converges for each \( x \in D \).

3b. What is a “diagonal process”? or “Cantor’s diagonal method”?

3c. State Ascoli-Arzela Theorem.

4. State and Prove Riesz representation theorem in Hilbert space.

5. Prove that every bounded sequence in a separable Hilbert space contains a weakly convergent subsequence. (Recall: Let \( S \) be a normed linear space. A sequence \(<x_n>\) is said to converge to \( x \in X \) if \( f(x_n) \to f(x) \) for all \( f \) in the dual space \( X^* \).)
6a. State the Radon-Nikodym Theorem.

6b. Let \((X, M)\) be a measurable space and let \(\mu, \nu, \lambda\) be finite measures defined on \(M\). If \(\nu \ll \mu\) and \(\mu \ll \lambda\) show that \(\nu \ll \lambda\) and that their corresponding Radon-Nikodym derivatives satisfy the following:

\[ \frac{d\nu}{d\lambda} = \left( \frac{d\nu}{d\mu} \right) \left( \frac{d\mu}{d\lambda} \right). \]

Let \((X, B, \mu)\) be a measure space.

7a. State the following theorems:
   (i) Lebesgue Dominated Convergence Theorem
   (ii) Fatou’s Lemma
   (iii) Monotone Convergence Theorem

7b. Prove that (ii) implies (i).

8a. Give an example of a function \(f : (0, 1) \to \mathbb{R}\) such that \(f \in L^p(0, 1)\) but \(f\) does not belong to \(L^q(0, 1)\) for some real number \(p\) and \(q\).

8b. Let \(f\) be a bounded measurable function on \([0, 1]\). Prove that \(\lim_{p \to \infty} ||f||_p = ||f||_\infty\).

9a. Define a (real-valued) function to be absolutely continuous on \([a, b]\).

9b. Define a (real-valued) function of bounded variation over \([a, b]\).

9c. Let \(f(x) = x^2 \sin(1/x)\) if \(x \neq 0, f(0) = 0\). Prove or disprove that \(f\) is of bounded variation in \([0, 1]\).

9d. Prove that if \(f\) is a Lipschitz on \((0, 1)\) (i.e. there exists a constant \(K\) such that \(|f(x_1) - f(x_2)| \leq K|x_1 - x_2|\) for every \(x_1\) and \(x_2\) in \((0, 1)\)), then it is differentiable almost everywhere.

10. Prove the following generalization of Fatou’s Lemma: if \(<f_n>\) is a sequence of non-negative functions, then

\[ \int \lim f_n \leq \lim \int f_n. \]