Work problems from each of the four groups I, II, III, IV as indicated. Throughout $X$ and $Y$ will denote arbitrary topological spaces and, unless otherwise specified, the real line $\mathbb{R}$ is assumed to be endowed with the Euclidean topology. Also, $\mathbb{Q}$ denotes the set of rational numbers, $J = \mathbb{R} - \mathbb{Q}$ and $\mathbb{C}$ is the set of complex numbers.

I. For the problems in this group, carefully define each topological term which appears, then verify the statement.

1. The collection $\mathcal{B} = \{ (a, b) \mid a, b \in \mathbb{R} \} \cup \{ \{ x \} \mid x \in J \}$ forms a basis for a topology on the real line $\mathbb{R}$.
2. A compact Hausdorff space is regular.
3. A contractible space $X$ has trivial fundamental group $\pi_1(X, x_0)$ for any $x_0 \in X$.

II. Work 4 of the 6 problems from this section.

4. Let $C$ be a component of $X$.
   a. Show that $C$ is closed.
   b. If $X$ is locally connected then $C$ is open.
   c. In general $C$ may not be open.
5. A function $f : X \to Y$ is continuous if and only if $f(A) \subset f(A)$ for each $A \subset X$.
6. Show that every nonconstant path in the Euclidean plane $\mathbb{R}^2$ contains a point one of whose coordinates is rational and a point one of whose coordinates is irrational.
7. Show that $\pi_1(X \times Y, x_0 \times y_0)$ is isomorphic to $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.
8. A connected metric space with more than one element is uncountable.

9. Let $X$ be the subspace of the Euclidean plane which is the union of $\{0\} \times [-1, 1]$ and $\{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 < 2\}$.
   a. Determine the fundamental group of $X$.
   b. Briefly explain why the “figure eight” is a deformation retract of $X$. 
III. Determine whether the following are true or false. In each case, provide either a proof or a counterexample as appropriate. (Be sure to explain the counterexample.)

10. The topology on $\mathbb{R}$ given in problem # 1 is metrizable.

11. Let $X$ be a first countable space and let $A$ be a closed subset of $X$.
   Then $x \in A$ if and only if there is a sequence $x_n$ in $A$ which converges to $x$.

12. Let $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$. The inclusion of $S^1$ into $\mathbb{C} - \{(\frac{1}{2},0)\}$ is a homotopy equivalence.

13. A locally compact Hausdorff space is regular.

IV. Work 2 of the 3 problems in this section.

14. A function $f : X \to \mathbb{R}$ is said to be lower semicontinuous if $f^{-1}((b, \infty))$ is open for each $b \in \mathbb{R}$. Let $\{ f_{\lambda} \mid \lambda \in \Lambda \}$ be a family of continuous functions from $X$ to $\mathbb{R}$ and suppose that $\{ f_{\lambda}(x) \mid \lambda \in \Lambda \}$ is bounded above for each $x \in X$. Define a function $M : X \to \mathbb{R}$ by $M(x) = \sup \{ f_{\lambda}(x) \mid \lambda \in \Lambda \}$.
   a. Show that $M$ is lower semicontinuous.
   b. Show that $M$ may fail to be continuous.

15. Let $X$ be a compact Hausdorff space. Suppose that $X$ has an open cover $\mathcal{U}$ where each $U \in \mathcal{U}$ is metrizable.
   a. Show that $X$ is metrizable.
   b. Show by example that if either the assumption that $X$ is compact or the assumption that each $U \in \mathcal{U}$ is open is removed then $X$ may fail to be metrizable.

16. Let $p : \tilde{X} \to X$ and $q : \tilde{Y} \to Y$ be covering maps.
   a. Show that $p \times q$ is a covering map from $\tilde{X} \times \tilde{Y}$ to $X \times Y$.
   b. Show that the group of covering transformations $A(\tilde{X} \times \tilde{Y}, p \times q)$ of $p \times q$ is isomorphic to $A(\tilde{X}, p) \times A(\tilde{Y}, q)$.
   c. The covering map $p \times q$ is regular if and only if both $p$ and $q$ are regular.