1. State and prove the Schroeder-Bernstein Theorem. Use it to prove that, if $X$ is an
infinite set, then $X$ has a proper subset of the same cardinality.

2. Describe the construction of the classical Cantor set $C$. Explain (with full jus-
ifications) how to construct a continuous monotone function $L$ on $[0,1]$ with
$L(0) = 0$, $L(1) = 1$ and which is constant on the complementary intervals of
$C$.

3. Let $S$ be an arbitrary set or real numbers. Define the outer measure $m^*(S)$. Prove
that, if $\{A_n\}_{n=1}^{\infty}$ is a countable collection of subsets of $\mathbb{R}$, then

$$m^* \left( \bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} m^*(A_n).$$

Explain what it means for the set $E \subset \mathbb{R}$ to be measurable. Prove that, if $\{E_n\}_{n=1}^{\infty}$
is a countable collection of disjoint measurable subsets of $\mathbb{R}$, then

$$m^* \left( \bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} m^*(E_n).$$

4. Find the values of the following limits, making clear and justified uses of appropriate
convergence theorems

$$\lim_{n \to \infty} \int_{0}^{1} \frac{nx \sin x}{1 + (nx)^2} \, dx \quad \quad \lim_{n \to \infty} \int_{0}^{\infty} \left( 1 + \frac{x}{n} \right)^{-n} \sin \frac{x}{n} \, dx.$$
5. State and prove Fatou’s Lemma. Give an example to show that strict inequality can occur.

6. Let the functions $f_\alpha$ be defined by

$$f_\alpha(x) = \begin{cases} 
  x \sin \frac{1}{x^\alpha}, & x > 0 \\
  0, & x = 0
\end{cases}$$

Find all the values of $\alpha \geq 0$ such that

a) $f_\alpha$ is continuous,

b) $f_\alpha$ is of bounded variation on the interval $[0, 1]$,

c) $f_\alpha$ is absolutely continuous on $[0, 1]$.