TOPOLOGY QUALIFYING EXAM
August 1994

Work as many problems as you can. Give complete explanations, but try not to waste time verifying obvious details.

1. Suppose \( f : X \to Y \) is a bijection, and \( f(A) = f(A) \) for every \( A \subset X \). Show that \( f \) is a homeomorphism.

2. Suppose \( X \) is compact and \( A \) is an infinite subset. Show directly from the definitions that \( A \) must have a limit point.

3. Let \( \mathbb{R}_+^2 = \{(x, y) : y > 0\} \), and set
   \[
   A = \{(x, y) \in \mathbb{R}_+^2 : (x, y) \text{ lies on a line of irrational slope passing through } (0, 0)\}
   \]
   \[
   B = \{(x, y) \in \mathbb{R}_+^2 : (x, y) \text{ lies on a line of irrational slope passing through } (1, 0)\}
   \]
   a. Show \( A \cup B \) is connected.
   b. Prove that every component of \( \mathbb{R}_+^2 \setminus (A \cup B) \) is a single point.

4. Let \( X \) be a Hausdorff space. Prove that \( X \) is normal if and only if given any closed subset \( F \) and any open set \( O \) such that \( F \subset O \), there exists an open set \( U \) such that \( F \subset U \subset O \).

5. Let \( X = \prod_{i=1}^{\infty}[0, 1] \), and let \( Y = \{x \in X \mid \pi_i(x) = 0 \text{ except for finitely many } i\} \).
   a. Determine whether \( Y \) is compact when \( X \) is given the product topology.
   b. Determine whether \( Y \) is compact when \( X \) is given the box topology.

6. Suppose \( X \) is normal, and \( A \subset X \) is closed.
   a. Prove that if \( f : A \to \mathbb{R}^n \), then \( f \) extends continuously to \( X \).
   b. Now suppose \( A \) is simply connected. Prove that if \( f : A \to S^1 \), then \( f \) extends continuously to \( X \). (Hint: What is the universal cover of \( S^1 \)?)

7. Let \( X \subset \mathbb{R}^2 \) be the set of vertical lines with integer \( x \)-intercepts.
   a. Describe the construction of \( \hat{X} \), the one-point compactification of \( X \), and explain its topology.
   b. Determine whether \( \hat{X} \) is homeomorphic to the Hawaiian earring, that is, the union of all circles in \( \mathbb{R}^2 \) with center \((1/n, 0)\) and radius \( 1/n \) where \( n \) is a positive integer.
   c. Determine whether \( \hat{X} \) is homeomorphic to \( \mathbb{R}/\mathbb{Z} \), that is, \( \mathbb{R} \) with \( \mathbb{Z} \) identified to a point.
Do two of the following three problems.

8. Let $X$ denote the space resulting from the identification of the edges of a two-dimensional hexagon as indicated in the diagram below. Use Van Kampen’s theorem to compute $\pi_1(X)$.

9. Let $C(Y,Z)$ be the space of continuous maps from $Y$ to $Z$. Given $f : X \times Y \to Z$, define $\tilde{f} : X \to C(Y,Z)$ by $\tilde{f}(x)(y) = f(x,y)$. Show that if $f$ is continuous then $\tilde{f}$ is continuous.

10. Let $Y$ be locally path connected. Suppose $f : X \to Y$ is a local homeomorphism and that $f$ has the path lifting property. Show that $f$ is a covering map.