Directions: Work as many problems as you can; you have three hours. You can appeal to results proved in your courses (unless you are asked to prove one of those results), but such results should be clearly cited. \( \mathbb{N}, \mathbb{Q}, \) and \( \mathbb{R} \) denote the positive integers, rational numbers, and real numbers, respectively.

1. (a) Let \( S \subseteq \mathbb{R} \) be closed, let \( f: S \to \mathbb{R} \) be continuous on \( S \), and let \((x_n) \subseteq S \) be a Cauchy sequence. Prove that \((f(x_n)) \) is a Cauchy sequence.
   (b) Show by example that the result of (a) can fail for continuous functions \( f: S \to \mathbb{R} \) if \( S \) is not closed in \( \mathbb{R} \).

2. Construct a closed set \( F \subseteq \mathbb{R} \) that has infinite Lebesgue measure, but contains no nonempty open intervals.

3. (a) Let \( M \) be a \( \sigma \)-algebra of subsets of a set \( X \), let \( Y \) be a set, and let \( f: Y \to X \) be a function. Prove that the family of sets
   \[ N = \{ f^{-1}(E) \mid E \in M \} \]
   is a \( \sigma \)-algebra on \( Y \).
   (b) Let \( f: \mathbb{R} \to \mathbb{R} \) be continuous; prove that for every Borel set \( Q \subset \mathbb{R} \) the inverse image \( f^{-1}(Q) \) is also a Borel set.

4. State and prove Egoroff’s theorem.

5. (a) Give a precise statement of the Monotone Convergence Theorem for the Lebesgue integral on an abstract measure space \( (X, M, \mu) \).
   (b) Consider the measure space \((\mathbb{N}, P(\mathbb{N}), \lambda)\), where \( P(\mathbb{N}) \) is the power set (i.e., set of all subsets) of \( \mathbb{N} \) and \( \lambda \) is the counting measure. If \( f: \mathbb{N} \to [0, \infty) \) is an arbitrary function, then give a precise proof that \( f \) is measurable and
   \[ \int_{\mathbb{N}} f \, d\lambda = \sum_{n=0}^{\infty} f(n), \]
   where the sum of the series is set to \( \infty \) if it diverges.

6. (a) Let \( (X, M, \mu) \) be a measure space and let \((f_n) \) be a sequence of nonnegative measurable functions of \( X \) into \([0, \infty)\) that converges in measure to a function \( f: X \to [0, \infty) \) and satisfies \( \lim_{n \to \infty} \int_X f_n \, d\mu = A \); prove that \( \int_X f \, d\mu \leq A \) (Hint: use Fatou’s lemma.)
   (b) Give an example where strict inequality holds in the conclusion of part (a).

7. If \( f, g: [a, b] \to \mathbb{R} \) are absolutely continuous functions on \([a, b]\), then prove that the product \( fg \) defined by \( (fg)(x) = f(x)g(x) \) is absolutely continuous on \([a, b]\).

8. (a) State Hölder’s inequality for functions defined on an abstract measure space \( (X, M, \mu) \).
   (b) Let \((X, M, \mu) \) be a measure space and let \( r, s \in \mathbb{R} \) be such that \( r > 1, s > 1 \). Find \( \alpha > 0 \) so that if \( f \in L^r(\mu) \) and \( g \in L^s(\mu) \), then \( |fg|^\alpha \in L^1(\mu) \).

9. Let \( g: \mathbb{R} \to \mathbb{R} \) be defined by \( g(t) = t^3 \), let \( h: \mathbb{R} \to \mathbb{R} \) be defined by \( h(t) = t^5 \), and let \( \lambda_g \) and \( \lambda_h \) denote the Lebesgue-Stieltjes measures associated to \( g \) and \( h \), respectively (view these measures as defined on the family \( B \) of Borel subsets of \( \mathbb{R} \)). Show that both of the relations \( \lambda_g \ll \lambda_h \) and \( \lambda_h \ll \lambda_g \) are true and compute the corresponding Radon-Nikodym derivatives \( d\lambda_g/d\lambda_h \) and \( d\lambda_h/d\lambda_g \).

10. Let \( (\mathbb{N}, P(\mathbb{N}), \lambda) \) be as in problem 5(b) and let \( ([0, 1], M, \mu) \) be the measure space where \( M \) is the family of Lebesgue measurable subsets of \([0, 1]\) and \( \mu \) is the Lebesgue measure. Define \( f: \mathbb{N} \times [0, 1] \to [0, \infty) \) by \( f(n, x) = (nx^{-n})/2^n \). Prove that \( f \) is measurable with respect to the product \( \sigma \)-algebra \( P(\mathbb{N}) \times M \) and evaluate
   \[ \int_{\mathbb{N} \times [0, 1]} f \, d(\lambda \times \mu); \]
   justify your steps and point out any standard theorems that you use.