0. INSTRUCTIONS

There are four parts to the exam. You are to work two, and only two, problems from each part. Each of the eight problems attempted will have equal weight and partial credit will be given.

I. GROUPS

Work two of the following problems.

1. Let $G$ be a simple group whose order is strictly greater than two. If $\phi: G \rightarrow S_n$ is a homomorphism of $G$ into the symmetric group of degree $n$, prove that $\text{Im}\phi$ is contained in the alternating group $A_n$. \textit{(Hint: It is the normality, not the simplicity, of $A_n$ which is relevant to this problem)}.

2. Let $k$ be a field, $G = GL_2(k)$. The sets $S = \{a \in GL_2(k) \mid \det(a) = 1\}$ and $\Delta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \mid 0 \neq \delta \in k \right\}$ are subgroups of $GL_2(k)$.
   (a) Define $\theta: \Delta \rightarrow Aut(S)$ by $\theta_d(a) = \theta_d(a) = dad^{-1}$. Prove that $\theta$ is a group homomorphism.
   (b) Prove that the semidirect product $S \rtimes_{\theta} \Delta$ is isomorphic to $GL_2(k)$.

3. Let $p$ and $q$ be distinct prime numbers. Prove that any group of order $pq$ is solvable.

II. RINGS

Work two of the following problems.

1. Let $R_1$ and $R_2$ be rings, $R = R_1 \times R_2$ their direct product. Prove that every ideal of the ring $R$ has the form $I = \{(a, b) \mid a \in I_1, b \in I_2\}$, where $I_1$ is an ideal of $R_1$ and $I_2$ is an ideal of $R_2$.

2. In the polynomial ring $\mathbb{Z}[X]$ let $f = X^3 - X + 2$.
   (a) Prove that $P = f\mathbb{Z}[X]$ is a prime ideal of $\mathbb{Z}[X]$.
   (b) Prove that $P$ is not a maximal ideal of $\mathbb{Z}[X]$.

3. Let $R$ be a commutative domain, $I$ an ideal of $R$. As is easily verified, the set $S = \{1 + a \mid a \in I\}$ is a multiplicatively closed subset of $R$. Prove that the extended ideal $S^{-1}I$ is contained in every maximal ideal of the localization $S^{-1}R$.
III. FIELDS
Work two of the following problems.

1. Let $f$ be a nonzero irreducible element in the polynomial ring $\mathbb{C}[x,y]$. If $F$ is the field of fractions of $\mathbb{C}[x,y]/(f)$, prove that the transcendence degree of $F$ over $\mathbb{C}$ is exactly one.

2. Let $E/k$ be a finite Galois extension, $G = \text{Aut}(E/k)$ the Galois group of the extension.
   (a) For each $\alpha \in E$, let $S_\alpha = \{ g \in G | g(\alpha) = \alpha \}$. Verify that $S_\alpha$ is a subgroup of $G$.
   (b) Now let $H$ be an arbitrary subgroup of $G$. Prove that $H = S_\alpha$ for some $\alpha \in E$.

3. Compute the Galois group over $\mathbb{Q}$ of the polynomial $X^6 - 3$.

IV. MODULES
Work two of the following problems.

1. Let $R$ be a principal ideal domain, $F$ a free $R$-module of finite rank. Let $\phi : F \to F$ be an $R$-endomorphism of $F$, $K = \text{Ker} \phi$. Prove that there exists an $R$-submodule $L$ of $F$ such that $K \oplus L = F$. (Hint: Be careful; “most” submodules of $F$ are not summands of $F$).

2. Let $R$ be a commutative ring, $I$ an ideal of $R$, $L = \{ a \in R | aI = 0 \}$.
   (a) Prove that each $a \in L$ induces an $R$-homomorphism $\lambda_a : R/I \to R$.
   (b) Using (a), prove that the $R$-modules $L$ and $\text{Hom}_R(R/I, R)$ are isomorphic.

3. Let $R$ be a commutative Noetherian ring, $R[X]$ the ring of polynomials over $R$, $I$ an ideal of $R[X]$. Prove that if $R[X]/I$ is a finitely generated $R$-module, then $I$ contains a monic polynomial.