MA/PhD qualifying examination in Topology. August, 1992

Instructions: Work as many problems as you can. Give clear and concise arguments; do not waste time by giving excessive detail. Apply major theorems when possible.

I. Let $X$ be a space and let $A$ and $B$ be connected subsets of $X$. Prove that if $A \cap B$ is nonempty, then $A \cup B$ is connected.

II. Let $X$ be a Hausdorff space. Prove that if every open subspace of $X$ is paracompact, then every subspace of $X$ is paracompact.

III. Let $z_n$ be a sequence of points in a product space $\prod_{\alpha \in A} X_\alpha$, and let $\pi_\beta: \prod_{\alpha \in A} X_\alpha \to X_\beta$ denote the projection map. Prove that the sequence $z_n$ converges if and only if for every $\beta \in A$, $\pi_\beta(z_n)$ converges in $X_\beta$.

IV. Let $X$ be a normal space and let $A = \{a_1, a_2, \ldots\}$ be a countable subset such that $a_i \neq a_j$ for $i \neq j$ and such that $A$ has no limit points in $X$.
   (a) Prove that the subspace topology on $A$ equals the discrete topology on $A$.
   (b) Prove that there exists an unbounded continuous function from $X$ to $\mathbb{R}$.

V. Let $f: [0, 1] \to S^1$ be defined by $f(t) = e^{6\pi it}$. Find explicitly the lift of $f$ to the standard covering space $p: \mathbb{R} \to S^1$ with initial point 2.

VI. Let $X$ be the quotient space obtained from the real line by collapsing the open interval $(0, 1)$ to a point. Prove that $X$ is not Hausdorff.

VII. Let $X$ be the one-point compactification of the integers. Construct an imbedding of $X$ into the real line $\mathbb{R}$.

VIII. Let $X$ be the reals with the lower limit topology. Prove that $X$ is separable but not second countable.

IX. Let $(X, d)$ be a metric space whose diameter equals 4. Let $U$ be the open cover $\{X - \{x\} \mid x \in X\}$. Prove that 3 is a Lebesgue number for $U$. 