Instructions: Work as many problems as you can. Give clear and concise arguments; do not waste time by giving excessive detail. Apply major theorems when possible.

I. Let $\prod X_\alpha$ be a product of a collection of nonempty spaces. Let $\pi_\beta: \prod X_\alpha \to X_\beta$ be the projection. Prove that $\pi_\beta$ takes open sets to open sets.

II. For $r \in \mathbb{R}$, define $A_r$ to be $\{r + n \mid n \in \mathbb{Z}\} \subseteq \mathbb{R}$. Let $U_r = \mathbb{R} - A_r$. Find an explicit locally finite open cover of $\mathbb{R}$ that is a refinement of $\{U_r \mid r \in \mathbb{R}\}$.

III. Prove that if $X$ is a regular space, $x \in X$, and $U$ is a neighborhood of $x$, then there exists a neighborhood $V$ of $x$ such that $V \subseteq U$.

IV. Let $I = [0, 1]$ and let $X = C(I, I)$, the space of continuous maps from $I$ to $I$. Prove that $X$ is not equicontinuous.

V. Recall that a subspace $A$ of $X$ is called a retract of $X$ if there exists a continuous function $r: X \to A$ such that $r(a) = a$ for all $a \in A$. Let $i: [0, 1] \to \mathbb{R}^n$ be an imbedding. Prove that $i([0, 1])$ is a retract of $\mathbb{R}^n$.

VI. Let $X$ be a Möbius band. Let $h: S^1 \to X$ be an imbedding which carries $S^1$ homeomorphically onto the boundary of $X$. Let $s_0 \in S^1$. What homomorphism is $h_*: \pi_1(S^1, s_0) \to \pi_1(X, h(s_0))$? Explain your answer.

VII. Let $X$ be a closed subset of a metric space $X$ and let $X^*$ be the quotient space obtained from $X$ by collapsing $A$ to a point. Prove that $X^*$ is Hausdorff.

VIII. Let $Y = \prod_{n=1}^{\infty} \mathbb{R}$ (with the product topology).

(a) Let $X_1 = \{(x_n) \in Y \mid \sum x_n \text{ converges and } x_n = 1\}$. Prove or disprove that $X_1$ is compact.

(b) Let $X_2 = \{(x_n) \in Y \mid \sum x_n^2 \text{ converges and } x_n^2 = 1\}$. Prove or disprove that $X_2$ is compact.

IX. Let $X = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of the natural numbers, let $Y$ be a metric space, and let $f: X \to Y$ be a function. Prove that $f$ is continuous if and only if the sequence $\{f(n)\}$ converges to $f(\infty)$.

X. Prove that if $f_0, f_1: X \to Y$ are homotopic maps, and $g_0, g_1: Y \to Z$ are homotopic maps, then $g_0 \circ f_0$ and $g_1 \circ f_1$ are homotopic maps. Prove that if $X$ is a space and $Y$ is a contractible space, then any two maps from $X$ to $Y$ are homotopic.