Do all problems from #1 – #6.

1. Definitions/Statements.
   a) Define: A topological space $X$ is **locally connected**:
   
   b) Let $X, Y$ be topological spaces; let $A$ be a subset of $X$. Let $f, g : X \to Y$ be two maps which coincide on $A$. Define what it means to say that the map $f$ is **homotopic to** $g$ relative to $A$:
   
   c) Describe a construction (without proof) of the **Stone-Čech compactification** $\beta X$ of a completely regular space $X$, and state its universal property (without proof).
   
   d) Let $X$ and $Y$ be topological spaces; and let $Y^X$ be the set of all maps from $X$ to $Y$. Define the **compact-open topology** on $Y^X$.
   
   e) State Urysohn’s Lemma:

2. Let $f : X \to Y$ be a continuous map. Prove or disprove: If $X$ is locally compact, then $f(X)$ is locally compact.

3. Let $X$ and $Y$ be topological spaces; let $F = \{ F_\alpha : \alpha \in J \}$ be a finite family of closed sets of a space $X$ which cover $X$. Let $f : X \to Y$ be a map whose restriction to each $F_\alpha$ is continuous. Prove or disprove: $f$ is continuous.

4. Let $X$ be a separable (i.e., $X$ contains a countable dense subset) regular space. Prove that every closed set is a $G_\delta$-set.

5. Prove that a closed surjective continuous map is a quotient map.

6. Let $X$ be a metric space; and let $A$ be an arbitrary subset. Recall $d(x, A)$, the distance from $x$ to $A$, is defined by $d(x, A) = \inf\{d(x, a) | a \in A\}$. Prove $\overline{A} = \{ y \in X | d(y, A) = 0 \}$.

Choose three problems from #7 – #10.

7. Let $p : \tilde{X} \to X$ be a covering space; let $\sigma : I \to X$ be a path. Suppose $f_0, f_1 : I \to \tilde{X}$ are lifts of the path $\sigma$ such that $f_0(0) = f_1(0)$. Prove: $f_0 = f_1$. Use “evenly covered neighborhoods” and compactness of $I$ (Uniqueness of Path Lifting).

8. Let $p$ and $q$ be two distinct points in the torus $S^1 \times S^1$. Let $X$ be the disjoint union of $S^1 \times S^1$ and the closed interval $I = [0, 1]$. Identify $p$ with $0 \in I$ and $q$ with $1 \in I$ to get a space $Y$. Compute the fundamental group of $Y$. Justify your answer.

9. Let $S^2$ be the standard 2-sphere and let $S^1$ be its equator. Prove that $S^1$ is not a retract of $S^2$.

10. Let $M$ and $N$ be surfaces with Euler characteristics $\chi(M)$ and $\chi(N)$, respectively. Calculate the Euler characteristic of the connected sum $M \# N$. 