Instructions: This is an open-book examination.

1. Let $f(x)$ be a continuous function from $[0, 1]$ to $[0, 1]$. Prove that there is an $x_0 \in [0, 1]$ such that $f(x_0) = x_0$.

2. Let $E$ be a Lebesgue measurable subset of $\mathbb{R}$ with $0 < m(E) < \infty$. Let $\chi_E$ denote the characteristic function of $E$.
   a) Prove that the function
   \[ \phi(x) = \int_{\mathbb{R}} \chi_E(y) \cdot \chi_E(x + y) dy \]
   is continuous at every point $x$ in $\mathbb{R}$.
   b) Use the result in part a) to show the following: Let $F = \{ x - y : x, y \in E \}$, there exists a $\delta > 0$ such that $(-\delta, \delta) \subset F$.

3. a) If $f(x)$ is of bounded variation on $[0, 1]$, show that for any $a \in (0, 1)$ the limit of $f(x)$ exists as $x \to a^-$.
   b) If $f(x)$ is absolutely continuous on $[0, 1]$, show that
   \[ T_0^1(f) = \int_0^1 |f'|. \]

4. Prove that $L^\infty[0, 1]$ is complete under $||\cdot||_\infty$. Does such a norm induce an inner product in $L^\infty[0, 1]$ (or: is $L^\infty[0, 1]$ a Hilbert space?)

5. Let $p \in [1, \infty)$ and $E$ be a measurable set in $\mathbb{R}$. Prove that $\lim_{n \to \infty} \int_E |f_n - f|^p = 0$ if and only if $f_n$ converges almost everywhere to a function $f \in L^p(E)$ and $\lim_{n \to \infty} \int_E |f_n|^p - \int_E |f|^p = 0$. 
We define \( l^2 \) space as the collection \( l^2 = \{ x = \{ x_i \}_{i=1}^\infty \mid \sum_{i=1}^\infty |x_i|^2 < +\infty \} \). We define the norm for an element \( x = \{ a_i \} \) in \( l^2 \) by

\[
\|x\|_2^2 = \sum_{i=1}^\infty |a_i|^2.
\]

It can be proved that \( l^2 \) is a Hilbert space.

A sequence \( \{ x^n \} \) in \( l^2 \) space is said to converge weakly to an element \( x \) in \( l^2 \) if for every \( b \in l^2 \)

\[
\langle x^n, b \rangle \rightarrow \langle x, b \rangle \quad \text{as} \quad n \rightarrow \infty,
\]

where \( \langle \cdot, \cdot \rangle \) is the inner product induced from the \( l^2 \) norm.

[10] a) Prove that if \( x^n \) converges to \( x \) in \( l^2 \) (usually we call such convergence “strongly convergent” comparing with “weakly convergent”), then \( x^n \) converges weakly to \( x \).

[10] b) Find a sequence that weakly converges to \( 0 \) in \( l^2 \), but does not strongly converge to \( 0 \).

[100] **Total Marks**