For each question, either supply a proof or counterexample. Giving a complete solution to one problem is better than two half solutions to two problems. Use a separate sheet for each problem.

(1) If \((X,d)\) is a complete metric space, recall that a map \(f : X \to X\) is called a contraction if there exists \(\alpha < 1\) s.t. \(d(f(x), f(y)) \leq \alpha d(x, y)\), for all \(x, y \in X\). Prove that if \(f\) is a contraction, then \(f\) has a unique fixed point in \(X\). What if \(\alpha = 1\) above? Is there still a fixed point?

(2) Let \(A\) and \(B\) be connected subsets of the topological space \(X\). Prove that if \(A \cap \overline{B} \neq \emptyset\), then \(A \cup B\) is connected.

(3) Let \(I = [0,1]\) and consider \(C(I, I)\), the space of continuous functions from \(I\) to \(I\). Is \(C(I, I)\) an equicontinuous family?

(4) Let \(X\) be the reals with the Lower limit topology. Prove that \(X\) has a countable dense subset, but does not have a countable basis.

(5) Let \(Y = \prod_{i=1}^{\infty} \mathbb{R}\) with the product topology. Consider the subset \(X = \{(x_n) \in Y : \sum x_n \text{ converge and } \sum x_n = 1\}\). Prove or disprove that \(X\) is compact.

(6) Prove that a connected manifold is path connected. (Recall: A manifold is a Hausdorff topological space which has a countable basis and is locally homeomorphism to an open subset of \(\mathbb{R}^m\)).

(7) Is the open ball centered at the origin in \(\mathbb{R}^n\) homeomorphic to \(\mathbb{R}^n\)? (If yes, exhibit a homeomorphism).

(8) Classify all the coverings of \(S^1\). Make sure you explicitly write down the coverings.

(9) Is there a retraction of \(S^2\) onto its equator?
(10) Let $G$ be a group of homeomorphisms of $S^n$. Assume $G$ is infinite. Can $G$ act properly discontinuously on $S^n$? If yes, give an example; if not, prove it.