ALGEBRA

Qualifying Exam

May 2000

Instructions: Do as many problems as you can and show all your work.

1. Let $G$ be a group of order 12. Show that either $G$ has a normal Sylow 3-subgroup or $G \cong A_4$.

2. Prove that if $H$ and $K$ are finite groups of $G$, whose orders are relatively prime, then $|HK| = |H| \cdot |K|$.
   (Here $HK = \{hk \mid h \in H, k \in K\}$.)

3.) Let $G$ be a group with $|G| = p^n$. Show for each $k \leq n$, $G$ has a normal subgroup of order $p^k$.

4.) Give an example of a ring with no maximal ideals.

5.) Show that if $R$ is a commutative ring with identity and $R[x]$ is a PID, then $R$ is a field.

6.) Let $A, B$ be $n \times n$ matrices over $\mathbb{Q}$. Suppose there is an invertible $n \times n$ complex matrix $C$ with $CAC^{-1} = B$. Prove that there is an invertible $n \times n$ rational matrix $D$ with $DAD^{-1} = B$.

7.) Classify, up to similarity, all $3 \times 3$ matrices $T$ satisfying $T^3 = T$ over $\mathbb{C}$.

8.) Let $K$ be the splitting field of $x^3 - 2$ over $\mathbb{Q}$. Determine all the intermediate fields $E$ between $K$ and $\mathbb{Q}$.

9.) Prove that it is impossible to construct the regular 9-gon by straightedge and compass.