Residues of Intertwining Operators for $SO^*_6$ as Character Identities

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Abstract

We show that the residue at $s = 0$ of the standard intertwining operator attached to a supercuspidal representation $\pi \otimes \chi$ of the Levi subgroup $GL_2(F) \times E^1$ of the quasisplit group $SO^*_6(F)$ defined by a quadratic extension $E/F$ of $p$-adic fields is proportional to the pairing of the characters of these representations considered on the graph of the norm map of Kottwitz-Shelstad. Here $\pi$ is self-dual, and the norm is simply that of Hilbert’s Theorem 90. The pairing can be carried over to a pairing between the character on $E^1$ and the character of $E^\times$ defining the representation of $GL_2(F)$ when the central character of the representation is quadratic, but non-trivial, through the character identities of Labesse-Langlands. If the quadratic extension defining the representation on $GL_2(F)$ is different from $E$ the residue is then zero. On the other hand when the central character is trivial the residue is never zero. The results agree completely with the theory of twisted endoscopy and $L$-functions and determines fully the reducibility of corresponding induced representations for all $s$.

1. Introduction

One of the major tools in representation theory of reductive groups, either over a local or a global field, is the theory of intertwining operators between parabolically induced representations. These are vector valued meromorphic functions which are basically a composite of functions of one variable. Harish-Chandra’s theory of $c$–functions, which is merely another name for intertwining operators, connects these important objects to both reducibility questions for the inducing spaces as well as Plancherel measures [10],[25].

On the other hand, the work developed by the first author (cf. [20]) connects these questions to arithmetic through $L$–functions [22] and their poles. In fact, one can use the information gathered from harmonic analysis (poles of intertwining operators) to define these $L$-functions and conversely [21]. Following [21], in a series of papers [6], [7], [8], Goldberg and the first author computed the residues at $s = 0$ for these operators in the cases where the group is a quasisplit classical group, the parabolic is maximal and the inducing data is supercuspidal. This gives the rank one setting necessary to determine the Plancherel measure and $R$-groups of Knapp-Stein and Harish-Chandra, as well as the $L$–functions (when the inducing data is generic).

One particularly important feature of this case is its connection with the theory of (twisted) endoscopy of Kottwitz-Shelstad [15] and Langlands, which reflects itself in the functorial transfer from quasisplit classical groups to $GL_n$ as established in [5] in the generic case and by Arthur in general (in progress). The corresponding $L$–functions are then those of Artin [11], [12], [26].

While significant progress was made in [6], [7], [8], the connection with endoscopy remained
conjectural. In fact, the residue was reduced to a sum of two terms $R_G$ and $R_{\text{sing}}$, with $R_G$ as an integral of products of twisted orbital integrals on $GL_n$ with orbital integrals on the classical group, related by the norm map, and $R_{\text{sing}}$ as a limit in the boundary [6], [7], [8]. But how the non-vanishing of either term implies the connection with the inducing data being transferred from each other, as predicted by endoscopy, was not clear.

On the other hand, the efforts of the second author who pursued the results in [6], [7], [8], with the goal of a precise interpretation of these residues, led to a very promising reformulation of these in [27].

The purpose of this paper is to completely verify these conjectures in a low-dimensional but important case. The results are under no assumptions since character identities needed for the agreement with what is predicted by the theory of $L$-functions as explained in [22] (Appendix to [27]).

Let $E/F$ be a quadratic extension of a $p$-adic local field of characteristic zero. Let $	ilde{G} = SO_6^*$ be the quasisplit special orthogonal group of rank 3 determined by $E/F$. Let $\tilde{B} = \tilde{T}\tilde{U}$ be the Borel subgroup of upper triangular elements in $\tilde{G}$, where $\tilde{T}$ is the Cartan subgroup of diagonals in $\tilde{B}$ and $\tilde{U}$ its unipotent radical. Let $M = GL_2 \times SO_2^*$ be the Levi subgroup of $\tilde{G}$ generated by the root $e_1 - e_2$ of $\tilde{T}$ (or $A_0$). If $P$ is the parabolic subgroup of $\tilde{G}$ with a Levi decomposition $P = MN$, $N \subseteq \tilde{U}$, then the simple root $\alpha$ in $N$ is simply the restriction of either $e_2 - e_3$ or $e_2 + e_3$ to $A_0$. We let $\tilde{\alpha} = (\rho_P, \alpha)^{-1} \rho_P$ as in [20], where $\rho_P$ is half the sum of roots in $N$.

Let $\pi$ be an irreducible supercuspidal representation of $GL_2(F)$ and $\chi_H$ a character of the torus $H(F) = SO_2^*(F) = E^1$, the subgroup of elements of norm one in $E^\times$. We are interested in

$$I(s\tilde{\alpha}, \pi \otimes \chi_H) = \text{Ind}_{M(F),N(F)}^{G(F)} \pi \otimes \chi_H \otimes q^{\langle s\tilde{\alpha}, H_M(\cdot) \rangle} \otimes 1,$$

where $s \in \mathbb{C}$. In particular, we would like to determine its points of reducibility by means of Plancherel measures [20]. This simply means to determine the poles of the intertwining operators $A(s\tilde{\alpha}, \pi \otimes \chi_H, w_0)$ at $s = 0$, where $w_0 = w_0^\pi \cdot w_{\tilde{\alpha}, \theta}^{-1}$ and $\theta = \{e_1 - e_2\}$.

The operator will have poles only if $w_0(\pi) \simeq \pi$, i.e., $\pi$ is self-dual or equivalently $\omega_\pi^2 = 1$, where $\omega = \omega_\pi$ is the central character of $\pi$. Consequently, $I(s\tilde{\alpha}, \pi \otimes \chi_H)$ will have reducibility points only if $\omega^2 = 1$.

When $\omega \neq 1$, then $\pi$ will be the Weil representation attached to $\text{Ind}_{W_{E^1}}^{W_{E'}} \chi'$, where $E'$ is a quadratic extension of $F$ and $\chi'$ is a character of $(E')^\times$. Moreover, if $G_+$ is the subgroup of $GL_2(F)$ consisting of elements $g$ for which $\text{det}(g)$ is a norm from $E'$, then the restriction $\pi|_{G_+}$ decomposes into the sum $\pi_+ \oplus \pi_-$ of two inequivalent representations. Write $\Theta_{\pi_\pm}$ for the character of $\pi_\pm$.

The main result of this paper is the following:

**Theorem 1.** The operator $A(s\tilde{\alpha}, \pi \otimes \chi_H)$ is holomorphic at $s = 0$, unless $\omega^2 = 1$.

a) Assume $\omega = 1$. Then $A(s\tilde{\alpha}, \pi \otimes \chi_H)$ has a simple pole at $s = 0$. (We assume the residual characteristic of $F$ is odd in this case, for simplicity.)

b) Assume $\omega^2 = 1$, but $\omega \neq 1$. Let $E'$ and $\chi'$ be as above. The residue of the operator $A(s\tilde{\alpha}, \pi \otimes \chi_H)$ at $s = 0$ is zero unless $E = E'$; set $\chi_G = \chi'$. In this case the residue is proportional to

$$\int_{E'} \chi_H(\gamma) \cdot \Delta_E(\hat{\gamma})(\Theta_{\pi_+}(\hat{\gamma}) - \Theta_{\pi_-}(\hat{\gamma}))d\gamma,$$
where $\gamma = \frac{\sigma(\tilde{\gamma})}{\tilde{\gamma}}$ is the norm of the element $\tilde{\gamma} \in E^\times$ through the norm map $\tilde{\gamma} \mapsto \gamma$ of Kottwitz-Shelstad which in this case is the map $F^\times \backslash E^\times \to E^1$ of Hilbert’s Theorem 90. Here $\Delta_E$ is the discriminant for $E^\times$ as a Cartan subgroup in $GL_2(F)$. In particular, using the character identities of Labesse-Langlands for the transfer $\chi_G \to \pi$, the residue is proportional to

$$\int_{E^1} \chi_H(\gamma) (\chi_G(\gamma) + \chi_G^{-1}(\gamma))d\gamma,$$

and therefore the residue is non-zero precisely when $\pi$ is attached to $Ind_{W_E}^W \chi_H$ as predicted by endoscopy and $L$-functions.

An immediate consequence of this theorem is the following reducibility criterion (cf. [20]):

**Proposition 1.** The induced representation $I(s\tilde{\alpha}, \pi \otimes \chi_H)$ is irreducible unless $\omega^2 = 1$.

a) Suppose $\omega = 1$. Then $I(\pi \otimes \chi_H)$ is irreducible. In this case $I(s^{\frac{1}{2}}\tilde{\alpha}, \pi \otimes \chi_H)$ is reducible and there are no other points of reducibility for $s \geq 0$.

b) Suppose $\omega^2 = 1$, but $\omega \neq 1$. Then $I(\pi \otimes \chi_H)$ is reducible unless $E = E'$ and $\pi$ is attached to $Ind_{W_E}^W \chi_H$. If $\pi$ is attached to $Ind_{W_E}^W \chi_H$, then $I(s\tilde{\alpha}, \pi \otimes \chi_H)$ is reducible and there are no other points of reducibility for $s \geq 0$.

**Corollary 1.** Our results are in complete agreement with those given in [22] using $L$-functions.

We remark that this paper gives a purely local proof of this result, whereas the method of $L$-functions is necessarily global.

The proof of the theorem is to apply the general formula obtained in [27] to this case. This is fairly non-trivial. The non-vanishing in the case $\omega = 1$ requires a bulky proof as one needs to use character values for $\pi$. In general, one clearly needs a more efficient way of proving the non-vanishing by relating the residue to the non-vanishing of the corresponding singular twisted orbital integral in [21] which gives the poles for the second $L$-functions $L(s, \pi, \Lambda^2)$ or $L(s, \pi, Sym^2)$, where $\pi$ is the representation on $GL_n(F)$.

We conclude our discussion of the case $\omega = 1$ by pointing out that in this case the residue becomes a pairing between the character of $\pi$, a representation of $GL_2(F)$, and $\chi_H$, one of $SO_2^2(F)$, which is not a natural one. This could justify the bulky calculations one has to deal with in proving the non-vanishing.

On the other hand, when $\omega \neq 1$, the pairing (1.1) comes out very naturally, and assuming the character identities generalizing those in [16], [18], (1.1) seems to be amenable to generalization. It seems that the residue formulas proven in [6],[7], [8], as reformulated in [27] are naturally suitable to detect the poles of the first $L$-function $L(s, \pi \times \chi_H)$ and their generalizations, rather than the second $L$-functions $L(s, \pi, \Lambda^2)$ or $L(s, \pi, Sym^2)$. In this paper, we have found that $R_{\text{sing}} = 0$, and the residue comes entirely from $R_G$. In cases of higher rank, we do not expect $R_{\text{sing}}$ to vanish.

Although the present work only deals with a low rank case, it still brings in many features of the general case, and is the first case where such subtle character identities appear so explicitly as a residue. With the reformulation presented in [27], and the present example complete, we plan to complete this project in our future work. In general one has a contribution to the residue for each maximal torus in $H$. One may treat the terms corresponding to compact tori through the methods of this paper. A more difficult matter is to study the weight factors of [27] which diverge for non-compact tori. We expect to apply a more delicate limiting process for these, and that weighted orbital integrals will play the role that ordinary orbital integrals do in the present paper. The result should again be a pairing of a twisted character values on $G$ with an ordinary characters value on $H$ at nonelliptic elements. One appeals to the theory of twisted endoscopic transfer in this situation;
if $\pi_G$ arises from $\pi_H$ via endoscopic transfer, this should point to the nonvanishing of the residue. This is work in progress.

We now describe the layout of this paper.

In Section 2, in addition to setting up notation, we describe the norm correspondence and compute its Jacobian, for our particular case. We emphasize that the calculation of measures through the Jacobian in particular is very delicate but vital for the matching that must occur with the character identities.

In Section 3, we recall the formula from [27] for the residue. In our case it simplifies considerably. The conclusion of this section is that the residue problem reduces to the integral

$$R(\pi, \chi_H) = \frac{1}{2 \log q} \int_{E^1} \chi_H(\gamma) \Theta_\pi^*(S(\gamma)^{-1})d^\ast\gamma.$$  

Here $\Theta_\pi^*$ is the twisted character associated to the self-dual representation $\pi$. The measure $d^\ast\gamma$ is proportional to $|\text{Tr}(\gamma) - 2\frac{1}{2}d\gamma|$, where $d\gamma$ is a normalized Haar measure on $E^1$.

The case of nontrivial central character is dealt with in Section 4. In this case the twisted character value is a difference of two character values, and lines up directly with the work of Langlands and Labesse as mentioned above.

Section 5 sets up the case of trivial central character. Here the twisted character value is merely the usual character value at $\gamma$. However the Jacobian weights the pairing to preclude the vanishing for any choice of $\chi_H$ and $\chi'$. Section 6 treats the case for which $E \neq E'$, and Section 7 treats the case for which $E = E'$. These sections provide a direct annulus computation using the character formulas of [23], [24].

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2. Preliminaries and Notation

2.1 The Group $\text{SO}_2$

Let $F$ be a $p$-adic field with ring of integers $O_F$, maximal ideal $p_F$, and residue field $k$ of order $q$. Let $G = \text{GL}_2(F)$. Write $w$ for the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For $g \in M_2(F)$, write $g^\ast$ for $w^t g w^{-1}$. Let $\varepsilon$ be the involution of $G$ given by $\varepsilon(g) = (g^\ast)^{-1}$. Say that elements $x, y \in G$ are $\varepsilon$-conjugate if there is a $g \in G$ so that $g x g^\ast = y$.

Fix a quadratic extension $E$ of $F$. Write $\sigma$ for the nontrivial automorphism of $E$, and $Nm$ and $\text{Tr}$ for the norm and trace maps from $E$ to $F$. Let $E^1$ denote the elements of $E$ of norm 1. We now specify a subgroup $H$ of $G$ isomorphic to $E^1$. Assume that the orthogonal form $J = \begin{pmatrix} 1 & 0 \\ 0 & -\tau \end{pmatrix}$ for some nonsquare $\tau \in O_F - p_F^2$. Then our group $H = \text{SO}_2 = \text{SO}(J)$ is given by matrices of the form

$$\begin{pmatrix} a & b \\ b\tau & a \end{pmatrix}$$  

(2.1)
corresponding to \(a + b\sqrt{\tau} \in E^1\). Note that if \(h \in H\) then \(h^+ = h\). Throughout this paper we will often identify \(E\) with the set of matrices of the form (2.1) with \(a, b \in F\). Write \(g_\theta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\); then the full orthogonal group is \(H^+ = O^+(J) = H \cup g_\theta H\). Note that \(g_\theta h g_\theta^{-1} = \sigma(h)\) for \(h \in H\). Since \(H\) is commutative in this case, we also write \(T = H\). Put \(T' = T - \{1\}\).

2.2 The Norm Correspondence for \(SO_6\)

Put

\[
\tilde{J} = \begin{pmatrix} J & w \\ w & J \end{pmatrix} \in \text{GL}_6(F),
\]

and \(\tilde{G} = SO(\tilde{J}) = \{g \in \text{SL}_6(F) | g \tilde{J} g = \tilde{J}\}\).

Let \(M\) be the Levi subgroup of \(\tilde{G}\) consisting of matrices of the form

\[
\begin{pmatrix} g & h \\ h & \varepsilon(g) \end{pmatrix},
\]

with \(g \in G\) and \(h \in H\). Write \(P\) for the parabolic subgroup generated by \(M\) and the Borel \(\tilde{B}\) of upper triangular matrices in \(\tilde{G}\). Then \(P = MN\), where \(N\) is the subgroup of matrices of the form

\[
n(X, Y) = \begin{pmatrix} I & X & Y \\ I & X' & Y \\ I & I & I \end{pmatrix}
\]

in \(\tilde{G}\). Here \(X, X', Y \in M_2(F)\). The condition that \(n(X, Y) \in \tilde{G}\) gives the equations

\[
X' = -J^t X w \quad \text{and} \quad Y + Y^+ = XX'.
\]

(2.2)

The integration over \(N\) defining our intertwining operator is decomposed into orbits via the action of the Levi subgroup \(M\) in [6]. The set of these orbits is equivalent to the set of \(\varepsilon\)-conjugacy classes of elements \(Y\) for which there is a solution to (2.2). We may parameterize these classes very simply by means of the \(\varepsilon\) norm correspondence from [6].

**Lemma 1.** Let \((X, Y)\) be a pair of invertible \(2 \times 2\) matrices satisfying (2.2), with \(Y\) invertible and put\(\text{Norm}(X, Y) = I - X'Y^{-1}X\).

i) We have \(\text{Norm}(X, Y) \in T'\).

ii) If \(g \in G\), then the pair \((g^{-1}X, g^{-1}Y\varepsilon(g))\) satisfies (2.2), and

\[
\text{Norm}(X, Y) = \text{Norm}(g^{-1}X, g^{-1}Y\varepsilon(g)).
\]

iii) If \(X\) is invertible, and \((X_1, Y)\) is also a solution to (2.2), then \(\text{Norm}(X_1, Y) = \text{Norm}(X, Y)\) or

\[
\text{Norm}(X_1, Y) = \text{Norm}(Xg_\theta, Y) = \text{Norm}(X, Y)^{-1}.
\]

**Proof.** It is straightforward to see that \(\text{Norm}(X, Y) \in H^+\). Since each \(\text{Norm}(X, Y) - I\) is invertible we obtain i). Part ii) is a computation.

The hypothesis for part iii) implies that \(XJ^t X = X_1J^t X_1\), which implies that \(X_1 = Xh\) with \(h \in H^+\). It follows that \(\text{Norm}(X_1, Y) = h^{-1} \text{Norm}(X, Y) h\), and the conclusion follows. \(\square\)

Write \(\mathcal{N}_\varepsilon\) for the set of \(\varepsilon\)-conjugacy classes \([Y]\) of invertible matrices \(Y\) for which Equations (2.2) have a solution for an invertible matrix \(X\). Write \(T'/\sigma\) for the quotient of \(T'\) under inversion (also conjugation, since \(T = E^1\)). Then Lemma 1 shows that \(\mathcal{N}_\varepsilon\) gives a well-defined map from \(\mathcal{N}_\varepsilon\) to \(T'/\sigma\),
given by \( N_\varepsilon([Y]) = \{\text{Norm}(X,Y), \text{Norm}(X,Y)^{-1}\} \). We will show this is a bijection. To understand the fibres of this map over \( T'/\sigma \), we relate it to the map \( \nu : G \to G \) given by \( \nu(g) = g^\sigma g^{-1} \).

**Lemma 2.** Suppose that \([Y_1], [Y_2] \in \mathcal{N}_\varepsilon \), and that \( N_\varepsilon([Y_1]) = N_\varepsilon([Y_2]) \). Then \([Y_2]\) contains an element \( Y_3 \) so that \( \nu(Y_1) = \nu(Y_3) \).

**Proof.** For \( i = 1, 2 \), pick invertible \( X_i \) satisfying condition (2.2) with \( Y_i \). Let \( \gamma_i = \text{Norm}(X_i,Y_i) \). The hypothesis implies that \( \gamma_2 = h\gamma_1 h^{-1} \), with \( h = 1 \) or \( h = g_\theta \).

By Lemma 3.3 of [6], we have
\[
X_i\gamma_i = -\nu(Y_i)X_i.
\]

Put \( Y_3 = (X_1h^{-1}X_2^{-1})Y_2(X_1h^{-1}X_2^{-1})^\sigma \); then a calculation shows that \( \nu(Y_1) = \nu(Y_3) \).

We therefore compute the fibres of the map \( \nu \).

**Lemma 3.** Fix a quadratic extension \( E \) over \( F \), and let \( g_1 \in E^\times \) with \( \sigma(g_1) \neq \pm g_1 \). Let \( \delta_1 = g_1g_\theta \) and \( \gamma = \nu(\delta_1) \). Then the fibre of \( \nu \) over \( \gamma \) is equal to \( Zg_1 \), where \( Z \) is the center of \( G \).

**Proof.** Note that \( \gamma \in E^1 - \{\pm 1\} \). Suppose that \( \delta_2 \in G \) with \( \nu(\delta_2) = \gamma \). This can be rewritten as \( \delta_2^\sigma = \gamma \delta_2 \). Since \( \gamma^\sigma = \gamma \), we obtain \( \delta_2 = \delta_2^\sigma \gamma \). Substituting, it follows that \( \delta_2 = \gamma \delta_2 \gamma \). Now let \( \delta_2 = g_2g_\theta \), for some \( g_2 \in G \). This gives \( g_2g_\theta = \gamma g_2g_\theta \gamma = \gamma g_2 \sigma(\gamma)g_\theta \). This implies that \( g_2 = \gamma g_2 \gamma^{-1} \), and therefore \( g_2 \in E^\times \). Now this gives
\[
\frac{\sigma(g_1)}{g_1} = \frac{\sigma(g_2)}{g_2} = \gamma,
\]
and it follows that \( \frac{g_1}{g_2} \in F^\times \), as desired.

Note that if \( z \in Z \), then \( g \) is \( \varepsilon \)-conjugate to \( z^2 g \). Actually, we know a little bit more:

**Lemma 4.** Let \( \delta \in E^\times \cdot g_\theta \), with \( \nu(\delta) = \gamma \in E^1 - \{\pm 1\} \), and \( \alpha \in F^\times \). Then \( \alpha \delta \) is \( \varepsilon \)-conjugate to \( \delta \) if and only if \( \alpha \) is a norm of \( E^\times \).

**Proof.** Suppose that \( \alpha = \text{Nm}(\beta) \) with \( \beta \in E^\times \). Then combining the facts that \( \beta^\sigma = \beta \), and that \( \beta g_\theta = g_\theta \sigma(\beta) \), we deduce that \( \beta \delta^\sigma = \alpha \delta \). For the other direction, suppose that
\[
g_\delta g_\delta = \alpha \delta \tag{2.3}
\]
and apply \( \nu \) to both sides. We obtain \( g \gamma g^{-1} = \gamma \), so that \( g \) commutes with \( \gamma \), and it follows that \( g \in E^\times \). Viewing \( g \) as an element of \( E^\times \) we write \( \sigma(g) \) for its conjugate. Now as above Equation (2.3) implies that \( g \sigma(g) \delta = \alpha \delta \), and therefore \( \alpha \) is the norm of \( g \).

In other words, there are two \( \varepsilon \)-conjugacy classes in the fibre of \( \nu \) over such \( \gamma \), corresponding to the two elements of \( F^\times / \text{Nm}(E^\times) \). However, we will soon see that there is only one \( \varepsilon \)-conjugacy class for the norm correspondence.

**Definition 1.** For \( \gamma \in T' \), we define \( S(\gamma) = wJ^{-1}(\gamma - 1) \).

The significance of \( S(\gamma) \) comes from the following proposition:

**Proposition 2.** The norm correspondence \( N_\varepsilon : \mathcal{N}_\varepsilon \to T'/\sigma \) is a bijection. More precisely, if \( \gamma \in T' \), then the fibre of the norm correspondence over \( \gamma \) is the singleton \([S(\gamma)^{-1}]\).

**Proof.** Let \( Y = S(\gamma)^{-1} = (\gamma - 1)^{-1}Jw^{-1} \). Then \( (I,Y) \) satisfies (2.2), and \( \text{Norm}(I,Y) = \gamma \), since \( Y + Y^\sigma = -Jw \) and \( I - I'Y^{-1} = \gamma \), as the reader may verify.
By Lemma 2, if \([Y_1] \in \mathcal{N}_r\) with \(N_\epsilon(Y_1) \ni \gamma\), then we may assume \(\nu(Y_1) = \nu(Y)\). Then by Lemma 3, there is an \(\alpha \in F^\times\) so that \(Y_1 = \alpha Y\). Therefore we must look for solutions \(X\) to \(\alpha X + \alpha Y = XX'\). This leads to the equation

\[\alpha J = XJXT^T;\]

in other words such an \(X\) exists if and only if the quadratic forms \(x_1^2 - \tau x_2^2\) and \(\alpha x_1^2 - \alpha \tau x_2^2\) are \(F\)-equivalent. Following [19], we compute the Hasse-Witt Invariants. The invariant for the first is 

\[(1, -\tau) = 1\]

and the invariant for the second is 

\[(\alpha, -\alpha \tau) = (\alpha, -\alpha)(\alpha, \tau) = (\alpha, \tau).\]

We are using here the Hilbert symbol and its elementary properties. Therefore such an \(X\) exists if and only if \(\alpha\) is a norm of \(E^\times\). But then by Lemma 4, \(\alpha Y\) is \(\epsilon\)-conjugate to \(Y\).

The element \(S(\gamma)^{-1}\) is not in \(E^\times\), but can be written in the form \(\tilde{\gamma}g_\theta\), with \(\tilde{\gamma} \in E^\times\). The relationship between \(\gamma\) and \(\tilde{\gamma}\) will play an important role in this paper, so we gather together a few properties. Let \(E = F[\sqrt{7}]\) be any quadratic extension of \(F\), with \(\tau\) a nonsquare in \(\mathcal{O}_E - p_E^2\).

Suppose \(\gamma = a + b\sqrt{7} \in E^1\), with \(a, b \in F\) and \(\gamma \neq \pm 1\). Throughout this paper we set

\[\tilde{\gamma} = \frac{\sqrt{7}}{1 - \gamma}.

**Lemma 5.** We have the following facts about \(\gamma\) and \(\tilde{\gamma}\).

i) \(\sigma(\gamma) = \sigma(\tilde{\gamma})\), \(\tilde{\gamma} - \sigma(\tilde{\gamma}) = \sqrt{7}\), and \(\frac{\sigma(\tilde{\gamma})}{\tilde{\gamma}} = \gamma\).

ii) We have \(\text{Nm}(\tilde{\gamma}) = \frac{\tau}{\text{Tr}(\tilde{\gamma}) - 2}\) and \(\text{Tr}(\tilde{\gamma}) = -\frac{b\tau}{a - 1}\).

For the next three items, suppose further that \(|\gamma - 1| < 1\), and let \(x = a - 1\).

iii) If \(\text{ord}(\tau) = 1\), then \(\text{ord}(x)\) is odd, and \(\text{ord}(x) = 2k + 1 \iff \text{ord}_E(\gamma - 1) = 2k + 1\).

iv) If \(\tau\) is a unit, then \(\text{ord}(x)\) is even, and \(\text{ord}(x) = 2k \iff \text{ord}_E(\gamma - 1) = k\).

v) There is a square root \(\lambda \in F^\times\) of \(\text{Nm}(\tilde{\gamma})\), so that \(\lambda^{-1} = (-b + \frac{x^2}{2a\tau}) \mod x^2\).

**Proof.** The first two are immediate. The condition \(|\gamma - 1| < 1\) implies that \(x \in p_F\). Since \(\gamma \in E^1\), we have

\[\tau b^2 = 2x + x^2.\]

Therefore we see that \(|x| = |\tau b^2|\), from which it follows that \(|\gamma - 1| = |\tau b^2|^{\frac{1}{2}} = |x|^{\frac{1}{2}}\). This gives (iii) and (iv). As for (v), it is straightforward to check that \(\text{Nm}(\tilde{\gamma})^{-1} = \frac{2x}{7}\) has a square root: one may divide both sides of (2.4) by \(\tau^{2\text{ord}(b) \tau}\) and then use Hensel’s Lemma. The rest is an easy calculation.

**2.3 Jacobians and Measures**

As we shall see later, to get the residues as an orthogonality relation as predicted in [22] when the central character is non-trivial, the Jacobians and measures must match precisely. The purpose of this section is to verify that this is in fact the case. We start by recalling the measures that show up in [6]. With notation as in Section 2.2 and the discussion in [6], the invariant measure for integration in formula (4.2) of [6] is

\[d^*(X, Y) = |\det Y|_F^{-(\rho_P, \tilde{\gamma})} d(X, Y).\]

Here \(d(X, Y)\) is the euclidean measure on \(N(F)\). For us, \(Y = \tilde{\gamma}g_\theta\) and \(d(X, Y) = dg d\tilde{\gamma}\).

We first recall that in the situation in hand, \(M\) is generated by \(e_1 - e_2\) and one can take either \(a' = e_2 - e_3\) or \(e_2 + e_3\) as the non–restricted root restricting to \(a\). Note that \(\rho_P = \frac{3}{2}(e_1 + e_2)\).
Then \( \langle \rho_P, \alpha \rangle = \frac{3}{2} \), so \( \tilde{\alpha} = \langle \rho_P, \alpha \rangle^{-1} \rho_P = e_1 + e_2 \). Consequently \( \langle \rho_P, \tilde{\alpha} \rangle = 3 \). (We remark that for \( GL_n \times SO_n^* \) as Levi in \( SO_n^* \), \( \langle \rho_P, \tilde{\alpha} \rangle = n - \frac{1}{2} \), when \( n > 2 \) is even). This gives

\[
d^*(X, Y) = |\det Y|_E^{-3}d(X, Y) = |\det(\tilde{\gamma})|_E^{-3}d\tilde{\gamma}.
\]

Now \( |\det(\tilde{\gamma})|_F = |Nm(\tilde{\gamma})|_E^{\frac{1}{2}} \), so by Lemma 5,

\[
|\det(\tilde{\gamma})|_E^{-3} = \left| \frac{\text{Tr}(\gamma) - 2}{\sqrt{\tau}} \right|_E^{\frac{3}{2}}.
\]

From the identity \( \tilde{\gamma}(1 - \gamma) = \sqrt{\tau} \) we obtain

\[
\frac{d\tilde{\gamma}}{d\gamma} = \left| \frac{\tilde{\gamma}}{1 - \gamma} \right|_E = \left| \frac{\sqrt{\tau}}{\text{Tr}(\gamma) - 2} \right|_E.
\]

For the rest of this paper, we will drop the subscript \( E \) from the norms.

**Definition 2.** Let \( d^* = |\tau|^{-1} \text{Tr}(\gamma) - 2|^{\frac{1}{2}}d\gamma \)

We are therefore able to write

\[
d^*(X, Y) = d^*\gamma dg.
\]

We need to compare this measure to other Jacobians which arise in the subject. Recall the definition of \( D_\varepsilon \):

**Definition 3.** Let \( \varepsilon \in G \). We write \( G_{\delta, \varepsilon} = \{ g \in G | g\delta g^\tau = \delta \} \). Let \( d\varepsilon : \text{Lie}(G) \to \text{Lie}(G) \) be the differential of \( \varepsilon \), given by \( d\varepsilon(X) = -X^\tau \). Then

\[
D_\varepsilon(\delta) = \det(\text{Ad}(\delta) \circ d\varepsilon - 1; \text{Lie}(G)/\text{Lie}(G_{\delta, \varepsilon})).
\]

**Proposition 3.** We have \( D_\varepsilon(\tilde{\gamma} g_\theta) = 2(\text{Tr}(\gamma) - 2) \).

**Proof.** Let \( G' = SL_2(F) \). It is easy to see that \( \text{Lie}(Z) \) is an eigenspace for \( \text{Ad}(\delta) \circ d\varepsilon - 1 \) with eigenvalue \(-2\); therefore we restrict our attention to \( \text{Lie}(G') \). Write \( \varepsilon_0(g) = \frac{1}{\det(g)}g \); recall that \( \varepsilon(g) = \text{Ad}(g_\theta) \circ \varepsilon_0 \). Therefore, on \( \text{Lie}(G') \), \( d\varepsilon \) is equal to \( \text{Ad}(g_\theta) \). We are reduced to computing \( \det(\text{Ad}(\tilde{\gamma}) - 1; \text{Lie}(G)/\text{Lie}(T)) \), where \( T \) is the usual centralizer of \( \tilde{\gamma} \). The matrix of the adjoint action of \( \tilde{\gamma} \) has eigenvalues \( \gamma \) and \( \sigma(\gamma) \). Therefore

\[
D_\varepsilon(\delta) = -2(\gamma - 1)(\sigma(\gamma) - 1)
= -2(\gamma\sigma(\gamma) - \gamma - \sigma(\gamma) + 1)
= 2(\text{Tr}(\gamma) - 2).
\]

It follows that

\[
d^*(X, Y) = |\tau|^{-1}|2|^{-\frac{1}{2}}|D_\varepsilon(S(\gamma))|^{\frac{1}{2}}d\gamma dg.
\]

Here is the definition of the discriminant \( \Delta_E \) mentioned in the introduction:

**Definition 4.** Let \( g \) be a matrix in \( \text{GL}_2(F) \) with distinct eigenvalues \( a, b \) in a quadratic extension \( E \). Then set

\[
\Delta_E(g) = \left| \frac{(a - b)^2}{ab} \right|^{\frac{1}{2}}.
\]
The following is straightforward.

**Proposition 4.** Let \( \gamma \in E^1 \), and \( \tilde{\gamma} = \left( \sigma(\gamma) - 1 \right) \sqrt{\tau} \). Then

\[
\Delta_E(\tilde{\gamma}) = |\operatorname{Tr}(\gamma) - 2|^\frac{1}{2} = \left| \frac{1}{2} D_\varepsilon(\tilde{\gamma}) \right|^\frac{1}{2}.
\]

So we may write \( d^*(X, Y) \) in a third way, as

\[
d^*(X, Y) = |\tau|^{-1} \Delta_E(\tilde{\gamma}) d\gamma dg.
\]

### 3. The Residue

**3.1 Recollection**

In this paper we are treating a low-dimensional example of a more general theory. Its history includes [6], [7], [8], which treat the quasisplit classical groups, in particular the case of even orthogonal groups. The present set-up may be generalized by letting \( \tilde{G} \) be an orthogonal group \( \text{SO}_{6n} \), and the Levi \( M = \text{GL}_{2n} \times \text{SO}_{2n} \) of three equal-sized blocks. Again one takes a self-dual supercuspidal representation \( \pi_G \) of \( G = \text{GL}_{2n}(F) \) and a supercuspidal representation \( \pi_H \) of \( H = \text{SO}_{2n}(F) \), and studies parabolic induction to \( \tilde{G} \) as in the introduction.

If the intertwining operator has a pole at zero, this pole will occur along a flat section of functions \( f \in I(s\pi_G \otimes \pi_H) \) assembled from some choice of matrix coefficients \( \psi \) of \( \pi_G \) and \( f_H \) of \( \pi_H \), and a pair of compact subsets \( L, L' \) of \( M_n(F) \). (See [6] for details.) Denote the central character of \( \pi_G \) by \( \omega \). Choose a compactly supported function \( f_G \) so that

\[
\psi(g) = \int_Z \omega(z)^{-1} f_G(zg) dz.
\]

The residue obtained from these choices is denoted \( R(f_G, f_H) \). The main result of [27] is the following theorem: The residue \( R(f_G, f_H) \) is equal to

\[
\operatorname{Res}_{s=0} \sum_{T} |W(T)|^{-1} \sum_{k=0}^{\infty} q^{-2nks} \int_{T} \sum_{S} \int_{G/T} \int_{T' \setminus H^+} f_G(gS(\gamma)^{-1} g) f_H(h^{-1} \gamma h) w_k(g, h) dh dg d^* \gamma.
\]

Here \( S \) runs over sections of the norm correspondence over \( \gamma \), and \( w_k(g, h) \) is a certain “weight function” defined in [27]. For us, the sum over \( S \) is a singleton by Proposition 2. Another sum runs over conjugacy classes of maximal tori \( T \) in \( H \), and \( |W(T)| \) denotes the order of the Weyl group of \( T \) in \( H \). By \( H^+ \) we denote the full orthogonal group \( O_{2n} \). A slight variation on the methods gives

\[
\operatorname{Res}_{s=0} \sum_{T} |W(T)|^{-1} \sum_{k=0}^{\infty} q^{-2nks} \int_{T} \int_{G/T} \int_{T' \setminus H^+} \sum_{\alpha \in A} \omega(\alpha)^{-1} f_G(\alpha g S(\gamma)^{-1} g) f_H(h^{-1} \gamma h) w_k(g, h) dh dg d^* \gamma,
\]

where \( w_k(g, h) = \operatorname{vol}(T \cap \omega^{-k} g^{-1} L h^{-1}) \) and \( A \) is a set of representatives for \( F^\times / F^\times 2 \).

In our case, where \( H = T = \text{SO}_{2n}^* \) and \( G = \text{GL}_2(F) \), a few simplifications occur. We have \( w_k(g, h) = w_k(g, 1) \), and so may write \( w_k(g) = w_k(g, h) \). Since \( T \) is compact and \( L \) contains a neighborhood of 0, we have \( \lim_{k \to \infty} w_k(g) = \operatorname{vol}(T) = 1 \) for all \( g \in G \). The quotient \( T' \setminus H^+ = H \setminus H^+ \) has order two; we write the quotient as \( \{ 1, g_\theta \} \). Write \( \gamma' = g_{\theta}^{-1} \gamma g_{\theta} \). Moreover we may take \( f_H \) to be a character \( \chi_H \) on \( H \). Write \( \pi \) for \( \pi_G \) and \( f \) for \( f_G \). The residue then simplifies to

\[
\operatorname{Res}_{s=0} \sum_{k=0}^{\infty} q^{-4ks} \int_{T} (\chi_H(\gamma) + \chi_H(\gamma')) \sum_{\alpha \in A} \omega(\alpha)^{-1} \int_{G/T} f(\alpha g S(\gamma)^{-1} g) w_k(g) dh dg d^* \gamma.
\]
Proposition 5. In this case, we have
\[ \lim_{k \to \infty} \int_T (\chi_H(\gamma) + \chi_H(\gamma')) \sum_{\alpha \in A} \omega(\alpha)^{-1} \int_{G/T} f(\alpha g S(\gamma) \gamma') w_k(g) dg^* \gamma = \]
\[ \int_T (\chi_H(\gamma) + \chi_H(\gamma')) \sum_{\alpha \in A} \omega(\alpha)^{-1} \int_{G/T} f(\alpha g S(\gamma) \gamma') dg^* \gamma. \]

Proof. The switching of limits follows from the usual reasoning: normalized twisted orbital integrals are bounded and have compact support on \( T \). The result then follows by Lebesgue’s dominated convergence theorem.

This combines nicely with the following elementary analysis.

Proposition 6. Let \( a_k \in \mathbb{C} \) be a sequence, and suppose \( \lim_{k \to \infty} a_k = a \). Then
\[ \lim_{s \to 0} s \sum_{k=0}^{\infty} a_k q^{-2nks} = \frac{a}{2n \log q}. \]

Proof. The change of variables \( x = q^{-2ns} \) reduces the problem to computing
\[ C \cdot \lim_{x \to 1} (\log x)^{\infty} \sum_{k=0}^{\infty} a_k x^k, \]
where \( C = -\frac{1}{2n \log q} \). By comparison with geometric series, we see the sum has radius of convergence at least one, thus converges absolutely for \( |x| < 1 \). So in this interval we may perform rearrangements. Since \( \lim_{x \to 1} \frac{\log x}{x-1} = 1 \), we may replace \( \log x \) with \( x - 1 \). Therefore we have
\[ (x-1) \sum_{k=0}^{\infty} a_k x^k = -a_0 + \sum_{k=0}^{\infty} (a_k - a_{k+1}) x^k \]
By Abel’s Limit theorem, this approaches \(-a\) as \( x \to 1^- \).

Corollary 2. The residue \( R(f_G, f_H) \) is independent of the choice of lattice \( L \) and equal to
\[ \frac{1}{4 \log q} \int_T (\chi_H(\gamma) + \chi_H(\gamma')) \sum_{\alpha \in A} \omega(\alpha)^{-1} \int_{G/T} f(\alpha g S(\gamma) \gamma') dg^* \gamma. \]

3.2 The Matrix Coefficient

Proposition 7. We have
\[ \sum_{\alpha \in A} \omega(\alpha)^{-1} \int_{G/T} f(\alpha g S(\gamma) \gamma') dg = \frac{1}{2} \int_{G/TZ} \psi(g S(\gamma) \gamma') dg. \]
Residues of Intertwining Operators

Proof. Putting \(\delta = S(\gamma)^{-1}\), this follows from the equalities

\[
\sum_{\alpha \in A} \omega(\alpha)^{-1} \int_{G/T} f(\alpha g \delta g^+) dg = \sum_{\alpha \in A} \omega(\alpha)^{-1} \int_{G/TZ} \int_{Z/\{\pm 1\}} f(\alpha z g \delta g^+) dz dg
\]

\[
= \frac{1}{2} \int_{G/TZ} \int_{Z} \sum_{\alpha \in A} \omega(\alpha)^{-1} f(\alpha z^2 g \delta g^+) dz dg
\]

\[
= \frac{1}{2} \int_{G/T} \int_{Z} \omega(z)^{-1} f(z g \delta g^+) dz dg
\]

\[
= \frac{1}{2} \int_{G/T} \psi(g \delta g^+) dg
\]

since \(\omega^2 = 1\) and \(T \cap Z = \{\pm 1\}\).

\[\square\]

Corollary 3. The residue \(R(f_G, f_H)\) is equal to

\[
\frac{1}{4 \log q} \int_T \chi_H(\gamma) \int_{G/Z} \psi(g S(\gamma)^{-1} g^+) dg d^* \gamma.
\]

Proof. A simple computation shows that \(S(\gamma)^{-1} = g_{\theta} S(\gamma)^{-1} g_{\theta}^+\), and therefore the orbital integral for \(\psi\) over the orbits of \(S(\gamma)^{-1}\) and \(S(\gamma')^{-1}\) agree. Moreover, since \(g_{\theta} w g_{\theta}^{-1} = -w\), we have \(g_{\theta} \varepsilon(g) g_{\theta}^{-1} = \varepsilon(g g g_{\theta}^{-1})\), which implies that \(\text{Ad}(g_{\theta}) \circ \varepsilon = \varepsilon \circ \text{Ad}(g_{\theta})\), and so

\[
\text{det}(\text{Ad}(g_{\theta} \delta g_{\theta}^{-1}) \circ \varepsilon - 1) = \text{det}(\text{Ad}(\delta) \circ \varepsilon - 1).
\]

In particular, \(D_{\varepsilon}(S(\gamma')) = D_{\varepsilon}(S(\gamma))\). This allows us to replace the factor \(\frac{\chi_H(\gamma) + \chi_H(\gamma')}{2}\) with \(\chi_H(\gamma)\). We may drop the quotient by \(T\) since it is compact with normalized measure. \(\square\)

At this point, we may discard some cases of nontrivial \(\omega\).

Proposition 8. If \(\omega\) restricted to the subgroup \(\text{Nm}(E^x)\) is nontrivial, then \(R(f_G, f_H) = 0\). Thus when \(\omega\) is nontrivial, we conclude that \(R(f_G, f_H) = 0\) unless \(E = E'\).

Proof. Let \(\delta \in E^x g_{\theta}\). Suppose that \(\beta \in E^x\), and that \(\omega(\beta \sigma(\beta)) = \omega(\alpha) \neq 1\). Then

\[
\int \psi(g \delta g^+) dg = \int \psi(g \beta \delta \beta^+ g^+) dg
\]

\[
= \int \psi(g \beta \sigma(\beta) \delta g^+) dg
\]

\[
= \int \psi(\alpha g \delta g^+) dg
\]

\[
= \omega(\alpha) \int \psi(g \delta g^+) dg,
\]

the integrals being over \(G/Z\). It follows that

\[
\int_{G/Z} \psi(g \delta g^+) dg = 0.
\]

\[\square\]

Definition 5. Let \(G\) be a \(p\)-adic reductive group, and \(\varepsilon : G \to G\) an involution. Suppose \((\pi, V)\) is an irreducible admissible representation of \(G\) with a nonzero intertwining operator \(I_{\varepsilon} : \pi \to \pi \circ \varepsilon\) satisfying \(I_{\varepsilon}^2(v) = v\).
Let $f \in C_c^\infty(G)$ and write $\pi^\varepsilon(f) : V \to V$ for the operator defined by
\[
\pi^\varepsilon(f)v = \int_{G/Z} f(x)\pi(x)I_\varepsilon v dx.
\]
By [4], there is a locally integrable function $\Theta^\varepsilon_\pi$ defined on the regular elements of $G$ so that for all such $f$,
\[
\text{Tr}(\pi^\varepsilon(f)) = \int_G \Theta^\varepsilon_\pi(x)f(x)dx.
\]

Note that if $I$ is any intertwining operator from $\pi$ to $\pi \circ \varepsilon$, then $I^2$ intertwines $\pi$. Therefore $I^2v = cv$ for some $c \in \C^\times$; by dividing by a square root of $c$ we obtain an involution $I_\varepsilon$ as above.

**Proposition 9.** Suppose in the above situation that $\pi$ is supercuspidal. Pick $v, \tilde{v} \in V$, and a $G$-invariant inner product $(,)$ which is also $I_\varepsilon$-invariant.

Let $\psi$ be the matrix coefficient defined by $\psi(g) = (\tilde{v}, gv)$. Let $x$ be a regular element of $G$. Then
\[
\int_{G/Z} \psi(gx\varepsilon(g)^{-1})dg = \Theta^\varepsilon_\pi(x)(\tilde{v}, I_\varepsilon v)d(\pi)^{-1},
\]
where $d(\pi)$ is the formal degree of $\pi$.

Note that if $(,)_1$ is a $G$-invariant inner product, then the inner product $(,)_2$ defined by $(v, w)_2 = (v, w)_1 + (I_\varepsilon v, I_\varepsilon w)_1$ is both $G$-invariant and $I_\varepsilon$-invariant.

We follow the proof of Theorem 9 in [9] closely.

**Proof.** Let $f \in C_c^\infty(G)$. We have
\[
\int_{G/Z} \psi(gx\varepsilon(g)^{-1})dg = \int_{G/Z} (g^{-1}\tilde{v}, x\varepsilon(g)^{-1}v)dg.
\]
Note that $\pi(\varepsilon(g)^{-1})v = I_\varepsilon \pi(g)^{-1}I_\varepsilon v$ for all $v$. Let $\{\phi_i\}_{i \in I}$ be an orthonormal basis of $V$. Then
\[
(g^{-1}\tilde{v}, x\varepsilon(g)^{-1}v) = \sum_{i,j \in I} (g^{-1}\tilde{v}, \phi_i)(\phi_i, xI_\varepsilon\phi_j)(\phi_j, g^{-1}I_\varepsilon v)
= \sum_{i,j} Q_{ij}(\tilde{v}, g\phi_i)(\overline{I_\varepsilon v, g\phi_j}),
\]
where $Q_{ij} = (\phi_i, \pi(x)I_\varepsilon\phi_j)$. So our orbital integral becomes
\[
\int_{G/Z} \sum_{i,j} Q_{ij}(\tilde{v}, g\phi_i)(\overline{I_\varepsilon v, g\phi_j})dg.
\]
Integrating this over $G$ against $f$ gives
\[
\int_G \int_{G/Z} \sum_{i,j} Q_{ij}(\tilde{v}, g\phi_i)(\overline{I_\varepsilon v, g\phi_j})f(x)dgdx = \int_{G/Z} \sum_{i,j} R_{ij}(\tilde{v}, g\phi_i)(\overline{I_\varepsilon v, g\phi_j})dg,
\]
where
\[
R_{ij} = \int_G (\phi_i, \pi(x)I_\varepsilon\phi_j)f(x)dx.
\]
By the orthogonality of matrix coefficients, Equation (3.1) becomes
\[
\left( \sum_i R_{ii} \right) d(\pi)^{-1}(\tilde{v}, I_\varepsilon v).
\]
Residues of Intertwining Operators

We note that

\[ \sum_i R_{ii} = \sum_i \int_G (\phi_i, \pi(x)I_x\phi_i) f(x) dx = \sum_i (\phi_i, \pi^\varepsilon(f)\phi_i) = \text{Tr} \pi^\varepsilon(f). \]

The conclusion follows, since

\[ \int_G \left( \int_{G/Z} \psi(gx\varepsilon(g)^{-1}) dg \right) f(x) dx = \text{Tr} \pi^\varepsilon(f)d(\pi)^{-1}(\tilde{v}, I_\varepsilon v). \]

\[ \square \]

Corollary 4. If we choose our \( \psi \) with \( v, \tilde{v} \) so that \((\tilde{v}, I_\varepsilon v) = d(\pi)\), then

\[ \int_{G/Z} \psi(gS(\gamma)^{-1}\varepsilon(g)^{-1}) dg = \Theta_\pi^\varepsilon(S(\gamma)^{-1}). \]

Let us keep this assumption on \( \psi \) throughout the paper, for simplicity. Another choice of \( \psi \) would involve multiplying the residue by a (possibly 0) scalar.

Returning to Corollary 3, note that \( D_\varepsilon(\alpha S(\gamma)) \) depends only on \( \text{Ad}(\alpha S(\gamma)) = \text{Ad}(S(\gamma)) \), so that \( D_\varepsilon(\alpha S(\gamma)) = D_\varepsilon(S(\gamma)) \). Moreover, \( \psi(g\alpha S(\gamma)^{-1}g^\gamma) = \omega(\alpha)\psi(gS(\gamma)^{-1}g^\gamma) \), where \( \omega \) is the central character of \( \pi \). We deduce the following theorem. Recall that \( d^*\gamma = |\gamma|^{-1} \text{Tr}(\gamma) - 2|\gamma|^2 d\gamma \), where \( d\gamma \) is the normalized Haar measure on \( T \).

Theorem 2. If \( \omega \) is trivial on the subgroup \( \text{Nm}(E^\times) \) of \( F^\times \), then

\[ R(f_G, f_H) = \frac{1}{2\log(q)} \int_T \chi_H(\gamma)\Theta_\pi^\varepsilon(S(\gamma)^{-1})d^*\gamma. \] (3.2)

Since the right hand side only depends on \( \pi \) and \( \chi_H \), we make the following definition.

Definition 6. Write \( R(\pi, \chi_H) \) for the right hand side of Equation (3.2).

We will compute \( R(\pi, \chi_H) \) in the sequel.

4. Case of Nontrivial Central Character

By Proposition 8, we may assume that \( E = E' \). To compute \( R(\pi, \chi_H) \), we need to find an intertwining operator \( I_\varepsilon \) as in Definition 5. Write \( \varepsilon_0(g) = \frac{1}{\det(g)}g \); recall that \( \varepsilon = \text{Ad}(g\theta)\circ\varepsilon_0 \). This means that \( \pi \circ\varepsilon_0(g) = \omega(\det(g))\pi(g) \). Moreover, since \( \pi \) is self-dual with a nontrivial central character, \( \omega \) is the sign character \( \text{sgn}_E \) associated with the extension \( E \).

The following is from Theorem 4.8.6 of [3]. Write \( G_+ \) for the subgroup of matrices in \( G \) whose determinant is a norm from \( E^\times \); it is a subgroup of \( G \) of index 2.

Proposition 10. The representation \( (\pi, V) \) is induced from a representation \( (\pi_+, V_+) \) of \( G_+ \).

The following lemma is familiar from Clifford theory.

Lemma 6. Let \( G \) be a group, and \( G_+ \) a subgroup of index 2. Pick an element \( s \in G - G_+ \). Let \( (\pi_+, V_+) \) be a representation of \( G_+ \). Write \( (\pi_-, V_-) \) for the \( G_+ \)-module whose representation space is again \( V_+ \) but where the action is given by \( \pi_-(g) = \pi_+(sgs^{-1}) \). Now let \( (\pi, V) = \text{Ind}_{G_+}^{G_+} \pi_+ \). Then \( V \) is isomorphic to \( V_+ \oplus V_- \) as a \( G_- \)-module. Let \( \chi \) be the nontrivial one-dimensional character of

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G which is trivial on \( G_+ \). Consider the operator \( I_\chi : V \to V \) which is the identity on \( V_+ \) and \(-1\) on \( V_- \). Then \( I_\chi \) is an intertwining operator from \( \pi_+ \) and \( \pi_- \). Here \( \chi \cdot \pi \) is the representation on \( V \) given by \( (\chi \cdot \pi)(g) = \chi(g)\pi(g) \).

We may also write \( I_\chi = P_+ - P_- \), where \( P_\pm \) is the projection from \( V \) to \( V_\pm \) orthogonal to \( V_\mp \).

**Proposition 11.** We have

\[
\Theta_\pi^e(S(\gamma)^{-1}) = \Theta_{\pi_+}(S(\gamma)^{-1}g_\theta) - \Theta_{\pi_-}(S(\gamma)^{-1}g_\theta),
\]

where \( \pi_- \) is the representation of \( G_+ \) given by \( \pi_- = \pi_+ (sgs^{-1}) \), where \( s \in G - G_+ \).

**Proof.** Pick an inner product on \( V \) which is \( G \) and \( I_\theta \)-invariant. By the above we may take \( I_\theta = \pi(g_\theta) \circ (P_+ - P_-) \). Let \( I_{\theta_0} = P_+ - P_- \); we will first compute the twisted character \( \Theta_{\pi_0}^e \) with respect to this intertwining operator. Note that \((,\) is also \( I_{\theta_0} \)-invariant.

Suppose \( f \in C_c(G) \) has support in \( G_+ \). For \( v \in V \), we have

\[
\pi_{\theta_0}^e(f)v = \int_G f(x)\pi(x)P_+vdx - \int_G f(x)\pi(x)P_-vdx.
\]

Let \( v^+ \in V_+ \) and \( v^- \in V_- \). Since \((,\) is \( I_{\theta_0} \)-invariant we have \((v^+, v^-) = (v^-, v^-) \), and therefore the vectors are orthogonal. Therefore there is an orthonormal basis of \( V \), written as a union \( \{e_i^+\} \cup \{e_i^-\} \), with \( e_i^+ \in V_+ \) and \( e_i^- \in V_- \). Then for all \( e_i^+ \) we have

\[
(\pi_{\theta_0}^e(f)e_i^+, e_i^+) = \left( \int_G f(x)\pi(x)e_i^+dx, e_i^+ \right)
\]

\[
= \left( \int_{G_+} f_+(x)\pi(x)e_i^+, e_i^+ \right)
\]

\[
= \left( \pi_+(f_+)e_i^+, e_i^+ \right),
\]

where we write \( f_+ \) for the restriction of \( f \) to \( G_+ \). Similarly, for all \( e_i^- \) we have

\[
(\pi_{\theta_0}^e(f)e_i^-, e_i^-) = - \left( \int_{G_+} f(x)\pi(x)e_i^-dx, e_i^- \right)
\]

\[
= - \left( \pi_-(f_+)e_i^-, e_i^- \right).
\]

It follows that \( \text{tr} \pi_{\theta_0}^e(f) = \text{tr} \pi_+(f_+) - \text{tr} \pi_-(f_+) \), and so for \( g \in G_+ \) we have

\[
\Theta_{\pi_0}(g) = \Theta_{\pi_+}(g) - \Theta_{\pi_-}(g).
\]

By translating by \( g_\theta \) we find that for \( g \in G_+g_\theta \), we have

\[
\Theta_\pi^e(g) = \Theta_{\pi_+}(gg_\theta) - \Theta_{\pi_-}(gg_\theta).
\]

The result follows since \( S(\gamma)^{-1}g_\theta \in E^x \subset G_+ \). \( \square \)

Let us summarize this as:

**Proposition 12.** Suppose that \( E = E' \), and the central character of \( \pi \) is \( sgn_E \). Then we have

\[
R(\pi, \chi_H) = \frac{1}{2\log q} \int_T \chi_H(\gamma) \left( \Theta_{\pi_+}(\gamma) - \Theta_{\pi_-}(\gamma) \right) d^*\gamma.
\]

Next, we turn to Lemma 7.19 in [18], where Langlands computes

\[
\Theta_{\pi_+}(\gamma) - \Theta_{\pi_-}(\gamma) = \pm \lambda(E/F, \psi) \text{sgn}_E \left( \frac{\gamma - \sigma(\gamma)}{\sqrt{T}} \right) \frac{\chi'(\gamma)}{\Delta_E(\gamma)}.
\]
Definition 5. We follow [23] with the following definitions.

5.1 Measures of Annuli

Proposition 2 of Shimizu [23] and Section 2.6 of Silberger [24]. But we set up the integration first.

Proposition 13. We have \( \chi'(\tilde{\gamma}) = \chi_G(\gamma) \) in the above situation.

Proof. In fact one has \( \alpha^\vee(\tilde{\gamma}) = \gamma^{-1} \) by Lemma 5.

Using Definition 2 and applying Proposition 4, we see that the discriminant terms cancel, and we obtain the following theorem.

Theorem 3. Suppose that \( E = E' \), and the central character of \( \pi \) is \( \text{sgn}_E \). Then \( R(\pi, \chi_H) \) is a nonzero constant multiple of \( \int_T \chi_H(\gamma) \left( \chi_G(\gamma) + \overline{\chi_G(\gamma)} \right) d\gamma. \)

Corollary 5. This verifies b) in Proposition 2 in [22].

5. Case of Trivial Central Character

Let us henceforth assume that the central character \( \omega \) of \( \pi \) is trivial. This is equivalent to the condition that the restriction of \( \chi' \) to \( F^\times \) is the sign character \( \text{sgn}_{E'} \) of \( F^\times \) associated to \( E' \). For the rest of this paper we restrict ourselves to the case of odd residual characteristic for simplicity. Again, our first step is to find an intertwining operator \( I_\varepsilon \) as in Definition 5. This is simpler than in the case of nontrivial \( \omega \); here \( \varepsilon = \text{Ad}(g_0) \circ \varepsilon_0 \). This means that \( \pi(\varepsilon(g)) = \pi(\text{Ad}(g_0)(g)) \), and therefore \( I_\varepsilon = \pi(g_0) \) intertwines \( \pi \) and \( \pi \circ \varepsilon \). We obtain the following proposition. Recall that \( \gamma = S(\gamma)^{-1} g_0. \)

Proposition 14. The residue \( R(\pi, \chi_H) \) is equal to

\[
\frac{1}{2 \log q} \int_T \chi_H(\gamma) \Theta_\pi(\gamma) d^* \gamma.
\]

For the values of \( \Theta_\pi \), we turn to the explicit character value computations which are in both Proposition 2 of Shimizu [23] and Section 2.6 of Silberger [24]. But we set up the integration first.

5.1 Measures of Annuli

We follow [23] with the following definitions.

Definition 7. For an integer \( n \geq 0 \), write

\[
C_n = \begin{cases} 
E^1 \cap (1 + p^2) & \text{if } E \text{ is ramified}, \\
E^1 \cap (1 + p^n) & \text{if } E \text{ is unramified, and } n \text{ is positive}, \\
E^1 & \text{if } E \text{ is unramified and } n = 0.
\end{cases}
\]

Write \( A_n \) for the “annulus” \( C_n - C_{n-1} \). Similarly, we write \( C'_n, A'_n \) for the corresponding subsets of \( E' \). For a nontrivial character \( \chi \) of the norm 1 group \( E^1 \) of a quadratic extension \( E \) of \( F \), write \( \ell_E(\chi) \) for the minimum \( n \) so that \( \chi \) is trivial on \( C_n \). We may drop the subscript \( E \) if it is understood. For us, \( \ell_E(\chi') \geq 1 \) since \( \chi' \) induces to a supercuspidal representation with trivial central character.

As in Section 2.3, we put \( d^* \gamma = |\tau|^{-1} |\text{Tr}(\gamma) - 2|^{1/2} d\gamma \). Much of the forthcoming integral computations reduce to the following lemmas.
Definition 8. Suppose $E$ is unramified over $F$, and $\chi$ is a character on $E^\times$ with $\ell = \ell_E(\chi)$. Then let

$$I(\ell, n) = \int_{C_n} \chi(\gamma) d^* \gamma.$$ 

The following follows from the usual annulus computation.

Lemma 7. 

$$I(\ell, n) = \begin{cases} 
- \frac{q^{-2\ell+3}}{(q+1)^2} & \text{if } n < \ell, \\
\frac{q^{-2n+2}}{(q+1)^2} & \text{if } n \geq \ell.
\end{cases} \quad (5.1)$$

This expression is nonzero. If we choose instead a ramified extension $E$, the result is proportional to this, and also nonzero.

Definition 9. Let $E$ be ramified, and $\chi$ a character on $E^\times$ with $\ell = \ell(\chi)$. Then let

$$I_r(\ell, n) = \int_{C_n} \chi(\gamma) d^* \gamma.$$ 

Lemma 8.

$$I_r(\ell, n) = \frac{q+1}{2\sqrt{q}} I(\ell, n).$$

6. Trivial Central Character, Different Tori

Theorem 4. Suppose that $E$ is not isomorphic to $E'$. Then $R(\pi, \chi_H) \neq 0$.

We will require some delicate information in the case when $E$ and $E'$ are both ramified but not isomorphic.

Definition 10. Let $E$ be a quadratic extension of $F$. Given $\gamma \equiv 1 \mod p$, let $g(\gamma) = \tilde{\gamma} \lambda$, where $\lambda$ is as in Lemma 5(v).

This is a rescaling of $\tilde{\gamma}$ so that $g(\gamma) \in E^1$.

Lemma 9. Suppose that $E = F[\sqrt{\tau}]$ is a ramified quadratic extension of $F$, with $\text{ord}(\tau) = 1$. Then

i) $\text{Tr}(g(\gamma)) \equiv (2 + \frac{x}{2}) \mod xb\tau$, where $\gamma = (1 + x) + b\sqrt{\tau} \in E^1$.

ii) The map from $C_n$ to $p_F^n$ taking $\gamma$ to $\gamma - \sigma(\gamma) \sqrt{\tau}$ induces an isomorphism of groups

$$B_n : C_n/C_{n+1} \rightarrow p_F^n/p_F^{n+1}.$$ 

Through this identification, we may view $C_n/C_{n+1}$ as a $k$-module. If $\mu \in k$, and $\gamma$ is in the quotient, we write this action as $\mu \gamma$.

iii) $\text{Tr}(\mu \gamma) - 2 \equiv \mu^2 (\text{Tr}(\gamma) - 2) \mod p_F^{2n+2}$.

iv) $\text{Tr}(g(\mu \gamma)) - 2 \equiv \mu^2 (\text{Tr}(g(\gamma)) - 2) \mod p_F^{2n+2}$.

Proof. Left to the reader. Note that if $a + b\sqrt{\tau} \in C_n$, then $x = a - 1 \in p_F^{2n+1}$ and $b \in p_F^2$. 

□

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DEFINITION 11. Let $E = F[\sqrt{\tau}]$ be a ramified extension of $F$, with ord$(\tau) = 1$. Write $\theta_n : p_F^n/p_F^{n+1} \to k$ for the isomorphism taking $b$ to $b/(-\tau)^n$. Write $\phi_n : C_n/C_{n+1} \to k$ for $\theta_n \circ B_n$.

Note that if $b \notin p_F^{n+1}$, then $\text{sgn}_E(b) = \text{sgn}_E(\theta_n(b))$ is equal to the Legendre symbol $(\frac{\theta_n(b)}{k})$ and $\phi_n(\sigma(g)) = -\phi_n(g)$. Also note that $\phi_n(1) \neq 0$ if $g \in A_n$. We now prove the theorem.

Proof. **CASE I:** $E'$ unramified and $E$ ramified.

Let $\gamma \in H = E^1$. As in [23], we take $E = F[\sqrt{-\varepsilon}]$. Let $\ell = \ell_E(\chi_H)$ and $\ell' = \ell_{E'}(\chi')$. We have

$$\Theta_\pi(\tilde{\gamma}) = \begin{cases} -2q^{\ell' - 1} & \text{if } \gamma \in C_{\ell' - 1}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, following Proposition 14, we obtain

$$R(\pi, \chi_H) = \frac{1}{2\log q} \int_{T}^{\chi_H(\gamma)\Theta_\pi(\tilde{\gamma})d^*\gamma} = -\frac{q^{\ell' - 1}}{\log q} \cdot I_r(\ell, \ell' - 1) \neq 0. \quad (6.1)$$

**CASE II:** $E'$ ramified and $E$ unramified.

This time we have

$$\Theta_\pi(\tilde{\gamma}) = \begin{cases} -(q + 1)q^{\ell' - 1} & \text{if } \gamma \in C_{\ell'}, \\ 0 & \text{otherwise.} \end{cases}$$

As in the previous proof we obtain

$$R(\pi, \chi_H) = -\frac{1}{2\log q} (q + 1)q^{\ell' - 1} \cdot I(\ell, \ell') \neq 0. \quad (6.2)$$

**CASE III:** $E'$ and $E$ both ramified and $\ell > \ell'$.

The convention is that $E' = F[\sqrt{-\varepsilon}]$ and $E = F[\sqrt{-\varepsilon_0}]$, where $\varepsilon_0 \in O_F^\times$ is a nonsquare. We write $\text{Tr}'$ for the trace map from $E'$ to $F$. Let $\gamma \in H = E^1$ and $\ell' = \ell_{E'}(\chi')$. We have

$$\Theta_\pi(\tilde{\gamma}) = \begin{cases} -(q + 1)q^{\ell' - 1} & \text{if } \gamma \in C_{\ell'}, \\ q^{\ell' - 1} \sum_{\beta \in C_{\ell' - 1}/C_{\ell'}} \chi'(\beta) \text{sgn}_{E'}(\text{Tr}(g(\gamma)) - \text{Tr}'(\beta)) & \text{if } \gamma \in A_{\ell' - 1}, \\ 0 & \text{otherwise.} \end{cases} \quad (6.3)$$

The new feature here is the sum over $\beta \in C_{\ell' - 1}/C_{\ell'}$. Since $\chi_H$ is nontrivial on $C_{\ell'}$, there is an element $\gamma_0 \in C_{\ell'}$ with $\chi_H(\gamma_0) \neq 1$. For all $\gamma \in C_{\ell' - 1}$ we have $\text{Tr}'(g(\gamma_0)) \equiv \text{Tr}'(g(\gamma)) \pmod{p^{2\ell}}$. We also have $\text{Tr}'(\beta) - 2 \in p^{2\ell' - 1}$ and $\text{Tr}(g(\gamma)) \in p^{2\ell' - 1} - p^{2\ell'}$ for $\gamma \in A_{\ell' - 1}$, and it follows that

$$\text{Tr}(g(\gamma)) - \text{Tr}'(\beta) \in p^{2\ell' - 1} - p^{2\ell'}.$$

Therefore we have

$$\int_{A_{\ell' - 1}} \chi_H(\gamma) \sum_{\beta \in C_{\ell' - 1}/C_{\ell'}} \chi'(\beta) \text{sgn}_{E'}(\text{Tr}(g(\gamma)) - \text{Tr}'(\beta))d\gamma =$$

$$\int_{A_{\ell' - 1}} \chi_H(\gamma_0) \sum_{\beta \in C_{\ell' - 1}/C_{\ell'}} \chi'(\beta) \text{sgn}_{E'}(\text{Tr}(g(\gamma)) - \text{Tr}'(\beta))d\gamma,$$
and it follows that this term is 0. As above we are done because
\[ R(\pi, \chi_H) = -\frac{1}{2\log q}(q + 1)q^{\ell - 1} \cdot I_r(\ell, \ell') \neq 0. \] (6.4)

**Case IV:** \( E' \) and \( E \) both ramified and \( \ell \leq \ell' \).

Equation (6.3) is still valid, but here the integral over \( A_{E'-1} \) will be nonzero. In this case \( \chi_H \) is trivial on \( C_{E'} \), and its restriction to \( C_{E-1} \) may be viewed as a character on \( C_{E'-1}/C_{E'} \). Note that \( \chi' \) restricts to a character on \( C_{E'-1}/C_{E'} \). Given \( \mu \in k \), write \( \chi^\mu \) for the character
\[ \chi^\mu(\beta) = \chi'(\mu \beta), \]
defined on this quotient. Of course, if \( \mu = 0 \), then \( \chi^0 \) is the trivial character.

Fix an element \( \gamma_0 = a_0 + b_0\sqrt{-\varepsilon_0} \in A_{E'-1} \), and let
\[ f(\beta) = \text{sgn}_{E'}(\text{Tr}(g(\gamma_0)) - \text{Tr}'(\beta)), \]
viewed as a function on \( C_{E'-1}/C_{E'} \). (In fact \( \text{Tr}(g(\gamma_0)) \neq \text{Tr}'(\beta) \) in all cases.) Write \( \hat{f}(\chi^\mu) \) for the Fourier coefficient of \( f \) with respect to the character \( \chi^\mu \). That is,
\[ \hat{f}(\chi^\mu) = \sum_{\beta \in C_{E'-1}/C_{E'}} \chi^\mu(\beta) \text{sgn}_{E'}(\text{Tr}(g(\gamma_0)) - \text{Tr}'(\beta)). \]

For \( \mu \neq 0 \), we have
\[ \hat{f}(\chi^\mu) = \sum_{\beta} \chi'(\beta) \text{sgn}_{E'}(\text{Tr}(g(\gamma_0)) - \text{Tr}'(\mu^{-1} \beta)). \]

But then by Lemma 9, we have
\[ \text{sgn}_{E'}(\text{Tr}(g(\gamma_0)) - \text{Tr}'(\mu^{-1} \beta)) = \text{sgn}_{E'}(\text{Tr}(g(\gamma_0)) - 2 + \mu^{-2}(2 - \text{Tr}'(\beta))) = \text{sgn}_{E'}(\text{Tr}(g(\mu \gamma_0)) - \text{Tr}'(\beta)). \]

Thus,
\[ \sum_{\beta \in C_{E'-1}/C_{E'}} \chi'(\beta) \text{sgn}_{E'}(\text{Tr}(g(\gamma)) - \text{Tr}'(\beta)) = \hat{f}(\chi^\mu), \] (6.5)
where \( \gamma = \mu \gamma_0 \). On the other hand,
\[ \hat{f}(\chi^0) = \sum_{\beta \in C_{E'-1}/C_{E'}} \text{sgn}_{E'}(\text{Tr}(g(\gamma_0)) - \text{Tr}'(\beta)) \]
\[ = \sum_{\beta} \text{sgn}_{E'} \left( \frac{x_0}{2} - 2y \right) \]
\[ = \sum_{\beta} \text{sgn}_{E'} (2x_0 - 8y). \]

Here we write \( \beta = c + d\sqrt{-\varepsilon} \), and put \( x_0 = a_0 - 1 \) and \( y = d - 1 \). This is equal to
\[ \sum_{\beta} \text{sgn}_{E'}(-\varepsilon_0 b_0^2 + 4\varepsilon d^2) = \sum_{\beta} \text{sgn}_{E'}(4d^2 - \varepsilon_0 b_0^2) \]
\[ = \sum_{\beta} \text{sgn}_{E'} \left( \frac{4d^2 - \varepsilon_0 b_0^2}{\varepsilon_0 2n} \right). \]

Now \( \frac{4d^2 - \varepsilon_0 b_0^2}{\varepsilon_0 2n} \) is a norm of \( F[\sqrt{\varepsilon_0}] \), and so it is a norm of \( E' \) if and only if it is a perfect square
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in $F$. As it is a unit, we may replace $\text{sgn}_{E'}$ with the Legendre symbol and obtain

$$\sum_{\mu \in k} \left( \frac{\mu^2 - \epsilon_0 \phi_{\ell-1}(\gamma_0)^2}{k} \right).$$

This is $-1$ (see Exercise 5.8 of [13]). Therefore we have

$$\hat{f}(\chi^0) = -1.$$

Consider the integral

$$\int_{A_{E'_{\ell-1}}} \chi_H(\gamma) \sum_{\beta \in C_{E'_{\ell-1}}/C_{E'}} \chi'(\beta) \text{sgn}_{E'}(\text{Tr}(g(\gamma)) - \text{Tr}'(\beta)) d\gamma.$$

Since $\chi_H$ is trivial on $C_{E'}$, this is equal to

$$\frac{1}{2} q^{-\ell'} \sum_{\gamma \in A_{E'_{\ell-1}}/C_{E'}} \chi_H(\gamma) \hat{f}(\chi^\mu).$$

(6.6)

Here $\mu = \mu(\gamma)$ is defined by $\gamma = \mu \gamma_0$. Fix an element $\beta_0 \in A_{E'_{\ell-1}}$. There is a $\mu_0 \in k$ so that

$$\chi_H(\gamma_0) = \chi^{\mu_0}(\beta_0).$$

So we may write (6.6) as

$$\frac{1}{2} q^{-\ell'} \sum_{\mu \in k} \chi^{\mu\mu_0}(\beta_0) \hat{f}(\chi^\mu) = \frac{1}{2} q^{-\ell'} \left[ -\hat{f}(\chi^0) + \sum_{\mu \in k} \chi^{\mu_0\mu}(\beta_0) \hat{f}(\chi^\mu) \right]$$

$$= \frac{1}{2} q^{-\ell'} (q \cdot f(\mu_0 \beta_0) + 1),$$

by Fourier inversion. We have

$$f(\mu_0 \beta_0) = \text{sgn}_{E'}(\text{Tr}(g(\gamma_0)) - \text{Tr}'(\mu_0 \beta_0)) = \pm 1,$$

with the sign depending on the relationship between $\chi'$ and $\chi_H$.

**Remark.** If $\ell < \ell'$, then $\mu_0 = 0$ and this is equal to $\text{sgn}_{E'}(2)$.

Putting this together, we obtain

$$\int_T \chi_H(\gamma) \Theta_\pi(\gamma) d^n \gamma = -(q + 1)q^{\ell'-1} I_\ell(\ell, \ell') + \frac{\sqrt{q}}{2} q^{-\ell' + 1} q^{-\ell'} (1 \pm q).$$

This is equal to

$$\frac{\sqrt{q}}{2} \left[ q^{-\ell'} + q^{-2\ell' + 1} (1 \pm q) \right],$$

and so we have the residue

$$R(\pi, \chi_H) = \frac{\sqrt{q}}{4 \log q} \left( q^{-\ell'} + q^{-2\ell' + 1} (1 \pm q) \right) \neq 0.$$  

(6.7)

7. **Trivial Central Character, Same Tori**

Next we handle the cases where $E = E'$. 

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Proof. By Lemma 5, we have \( \sigma(\overline{\gamma}) = \gamma \). By Lemma 5(iv) either \( \text{Nm}(\overline{\gamma}) \) is a square in \( F^\times \) or the product of a square and \( \tau \). In the first case, we may write
\[
\chi(\overline{\gamma}) = \chi \left( \frac{\overline{\gamma}}{\sqrt{\gamma} \sigma(\gamma)} \right) \cdot \text{sgn}_E(\sqrt{\text{Nm}(\overline{\gamma})})
\]
\[
= \chi \left( \frac{\overline{\gamma}}{\sqrt{\sigma(\gamma)}} \right) \text{sgn}_E(b)
\]
\[
= \chi(\sqrt{\gamma^{-1}}) \text{sgn}_E(b).
\]
Similarly, in the second case, \( \chi(\overline{\gamma}) = \chi(\sqrt{\gamma^{-1}\tau}) \text{sgn}_E(b) \). Therefore, the proposition holds with
\[
\chi_G(\gamma) = \begin{cases} 
\chi'(\sqrt{\gamma}) & \text{if } \gamma \text{ is a square,} \\
\chi'(\sqrt{\gamma/\tau}) & \text{if } \gamma \tau \text{ is a square.}
\end{cases}
\]
This defines a character on \( E^1 \).

Remark. In fact, if \( \ell' = \ell_E(\chi') \geq 1 \) as in our case, then \( C_{\ell}^2 = C_{\ell'} \), and it follows that \( \ell_E(\chi_G) = \ell_E(\chi') \). Also note that \( \chi_G(\gamma)^2 = \chi'(\gamma) \) for all \( \gamma \in E^1 \).

Theorem 5. Suppose that \( E = E' \) is the unramified quadratic extension of \( F \). Then \( R(\pi, \chi_H) \neq 0 \).

Proof. Let \( \chi_G \) be as in Proposition 15, applied to \( \chi' \). Let \( \ell' = \ell_E(\chi') \) and \( \ell = \ell_E(\chi_H) \). We have by [23],
\[
\Theta_\pi(\overline{\gamma}) = \begin{cases} 
-2q^{\ell'-1} & \text{if } \gamma \in C_{\ell'}, \\
(-1)^{\ell'} |\text{Tr}(\gamma) - 2|^{\frac{1}{2}}(\chi_G(\gamma) + \overline{\chi_G(\gamma)})(-1)^{\text{ord}(\gamma^{-1})} & \text{otherwise.}
\end{cases}
\]

Our integral \( \int_{\Gamma} \chi_H(\gamma)\Theta_\pi(\overline{\gamma})d^s\gamma \) is equal to
\[
(-1)^{\ell'} \sum_{k=0}^{\ell'-1} (-1)^k \int_{A_k} \chi_H(\gamma)(\chi_G(\gamma) + \overline{\chi_G(\gamma)})d\gamma - 2q^{\ell'-1}I(\ell, \ell'). \tag{7.1}
\]

Write \( T_k \) for \( \int_{A_k} \chi_H(\gamma)(\chi_G(\gamma) + \overline{\chi_G(\gamma)})d\gamma \). Note that each \( T_k \) is a rational number by orthogonality, since \( \text{vol}(A_k) \) is rational. Also \( I(\ell, \ell') \) is rational by Lemma 7, and therefore Expression (7.1) is a rational number. We simply wish to show that it is nonzero. Pick a prime number \( r \) dividing \( q + 1 \), and say \( \text{ord}_r(q + 1) = e \). Then \( \text{ord}_r(I(\ell, \ell')) = -2e \) but \( \text{ord}_r(T_k) \geq -e \) for all \( k \). Suppose \( r \neq 2 \). Then Expression (7.1) has \( \text{ord}_r = -2e \) and is therefore nonzero. (The case in which \( q + 1 \) is a power of 2 is similar but left to the reader.)

We will do something similar for the ramified case, but it is a little more complicated. Let \( E^\times \) be a ramified extension of \( F \). By Hensel’s Lemma, any element \( x \in 1 + p_E \) has a unique square root in \( 1 + p_E \). It follows that there is a well-defined square root function \( r : C_0 \to C_0 \), so that \( r(\gamma^2) = \gamma \) for \( \gamma \in C_0 \).

Proposition 16. Let \( E \) be ramified over \( F \). Suppose that \( \chi : E^\times \to C^\times \) is a linear character whose restriction to \( F^\times \) is \( \text{sgn}_E \). Then

\[
\chi_G(\gamma^{-1}) \text{sgn}_E(b) = \chi'(\overline{\gamma}).
\]

As usual \( \gamma = a + b\sqrt{\tau} \).
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i) If \( \gamma \in E^1 - C_0 \), then \( \chi(\tilde{\gamma}) = \text{sgn}_E(2)\chi(\sqrt{\tau})(\chi \circ r)(-\gamma^{-1}) \).

ii) If \( \gamma \in C_0 \), then \( \chi(\tilde{\gamma}) = \text{sgn}_E(-b)(\chi \circ r)(\gamma^{-1}) \).

Proof. As usual, write \( \gamma = a + b\sqrt{\tau} \). Then \( \text{Nm}(\tilde{\gamma}) = \frac{r}{2a - 2} \equiv -\frac{r}{4} \mod \mathfrak{p} \). Let \( \lambda \in F^\times \) be the square root of \( -\frac{\text{Nm}(\tilde{\gamma})}{\tau} \) which is congruent to \( \frac{1}{2} \mod \mathfrak{p} \). Note that

\[
\left( \frac{\tilde{\gamma}}{\lambda\sqrt{\tau}} \right)^2 = -\frac{\tilde{\gamma}^2}{\lambda^2\sigma(\tilde{\gamma})} = -\gamma^{-1}.
\]

A computation shows that \( \frac{\tilde{\gamma}}{\lambda\sqrt{\tau}} \equiv 1 \mod \mathfrak{p} \), and it follows that

\[
\frac{\tilde{\gamma}}{\lambda\sqrt{\tau}} = r(-\gamma^{-1}).
\]

Therefore

\[
\chi(\tilde{\gamma}) = \text{sgn}_E(\lambda)(\sqrt{\tau})\chi(r(-\gamma^{-1})),
\]

as desired. The second part is easier (use Lemma 5.5). \( \square \)

Gauss sums make an appearance in the ramified case.

Definition 12. Let \( \chi : E^1 \rightarrow \mathbb{C}^\times \) be a character, with \( \ell = \ell_E(\chi) \geq 1 \). Let \( \Lambda_\chi : k \rightarrow \mathbb{C}^\times \) be the additive character given by \( \chi \circ \phi_{\ell^{-1}} \), where \( \phi_{\ell^{-1}} \) is given in Definition 11. We put

\[
\tau(\chi) = \sum_{a \in k} \left( \frac{a}{\ell} \right) \Lambda_\chi(a) \quad \text{and} \quad \varepsilon(\chi) = \frac{\tau(\chi)}{\sqrt{q}}.
\]

It is well-known (see [13]) that \( \varepsilon(\chi)^2 = (-\frac{q}{\ell}) \).

Theorem 6. Suppose that \( E = E' \) is a ramified quadratic extension of \( F \), and the central character of \( \pi \) is trivial. Then \( R(\pi, \chi_H) \neq 0 \).

Proof. Let \( \ell' = \ell_E(\chi') \) and \( \ell = \ell_E(\chi_H) \). The convention is that \( E = F[\sqrt{-\omega}] \). We have by [23],

\[
\Theta_\pi(\tilde{\gamma}) = \begin{cases} 
-(q + 1)q^{\ell'-1} & \text{if } \gamma \in C_{E'}, \\
q^{\ell'-1} \sum_{\beta} \chi'(\beta) \text{sgn}_E(Tr(g(\gamma) - \beta)) & \text{if } \gamma \in A_{E'} \\
\varepsilon(\chi')(\chi')^{-1} \varepsilon(\gamma) = \frac{\tau(\chi)}{\sqrt{q}} & \text{otherwise}.
\end{cases}
\]

The sum is over \( \beta \in C_{E'}/C_E \) with \( \beta \neq g(\gamma), g(\sigma(\gamma)) \) mod \( C_E \). Here \( \varepsilon(\chi') \) is the root of unity attached to \( \chi' \) as in [23]. Also let \( \tau(\chi') = \varepsilon(\chi')\sqrt{q} \). Put \( \chi_1 = \chi_H \circ (\chi \circ r)^{-1}, \chi_2 = \chi_H \circ (\chi' \circ r)^{-1}, \) and \( \ell_i = \ell(\chi_i) \). Using Proposition 16 and Fourier inversion on \( C_{E'}/C_E \) again, the integral \( \int_T \chi_H(\gamma)\Theta_\pi(\tilde{\gamma})d^*\gamma \) is equal to the sum of the three terms

\[
\begin{cases} 
\left( \frac{\sqrt{q}}{2} \right) q^{-2\ell + \ell' + 1} & \text{if } \ell' < \ell \\
\left( \frac{\sqrt{q}}{2} \right) q^{-\ell'} & \text{if } \ell' \geq \ell,
\end{cases}
\]

\[
\begin{cases} 
0 & \text{if } \ell' < \ell \\
\left( \frac{\sqrt{q}}{2} q^{-\ell'}(1 + q) & \text{if } \ell' \geq \ell, and
\end{cases}
\]

\[
\frac{1}{2} \varepsilon(\chi')^{-1} \left( q - 1 \right) \left[ \text{sgn}_E(-1)\tau(\chi_H)(q^{-\ell_1}) \right]_1 + \left[ \tau(\chi_2)q^{-\ell_2} \right]_2 + \left[ \pm q \right]_3.
\]

Here the \([ ]_1\) term only appears if \( \ell_1 < \ell' - 1 \), the \([ ]_2\) term only appears if \( \ell_2 < \ell' - 1 \), and the \([ ]_3\) term only appears if \( \ell_1 \) or \( \ell_2 = 0 \). Of these, it is clear that ord_\_q of (7.4) is \( \geq -\ell' + 1 + \frac{1}{2} \). Term (7.2)
has ordₚ ⩽ −ℓ + ½. If ℓ ⩽ ℓ', then ordₚ of (7.3) is also −ℓ' + ½, but is clear that the sum of (7.2) and (7.3) will not have a lower ordₚ than (7.2). Therefore the sum cannot be zero, concluding the proof.

Corollary 6. This agrees with a) in Proposition 2 in [22].

Theorem 1 is now proved.

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