Math 6393: Lie Groups and Lie Algebras II

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— 3rd Assignment —

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8. Let V be a finite-dimensional vector space over a field k. Let V^* be the dual space. For an endomorphism A of V, let $A^{\rm tr}$ be the dual endomorphism of V^*.

i) Show that there is a *canonical* isomorphism of k vector spaces $(V^*)^* \cong V$.

ii) Let $v_1, \ldots, v_n$ be a basis of V, and let $\phi_1, \ldots, \phi_n$ be the dual basis of V^*. Let M be the matrix of A with respect to $v_1, \ldots, v_n$. Show that the matrix of $A^{\rm tr}$ with respect to $\phi_1, \ldots, \phi_n$ is the transpose of M.

9. Let Π be a finite-dimensional representation of a Lie group G. Let Π^* be the dual representation.

i) Show that there is a *canonical* equivalence of representations $(\Pi^*)^* \cong \Pi$.

ii) Show that Π is irreducible if and only if Π^* is irreducible.

10. Let

$$E_1 = \frac{1}{2}\begin{bmatrix} i & \\ & -i \end{bmatrix}, \qquad E_2 = \frac{1}{2}\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, \qquad E_3 = \frac{1}{2}\begin{bmatrix} & i \\ i & \end{bmatrix}$$

and

$$F_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \qquad F_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then E_1, E_2, E_3 is a basis of $\mathfrak{su}(2)$, and F_1, F_2, F_3 is a basis of $\mathfrak{so}(3)$. Identify the vector space $\mathfrak{su}(2)$ with $\mathbb{R}^3$ by identifying E_1, E_2, E_3 with the standard basis vectors of $\mathbb{R}^3$. Then ${\rm ad}_{E_i}$ becomes an endomorphism of $\mathbb{R}^3$.

i) Show that ${\rm ad}_{E_i} = F_i$ for $i = 1, 2, 3$.

ii) Now consider ${\rm Ad} : {\rm SU}(2) \to {\rm GL}(\mathfrak{su}(2)) \cong {\rm GL}(3, \mathbb{R})$. Show that the image of Ad is SO(3), and that the kernel of Ad is $\{\pm I\}$.