Math 6393: Lie Groups and Lie Algebras II

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— 1st Assignment —

due Jan. 22, 2008

1. Let G be a connected matrix Lie group, let Π_1 and Π_2 be representations of G, and let π_1 and π_2 be the associated Lie algebra representations. Prove that π_1 and π_2 are equivalent if and only if Π_1 and Π_2 are equivalent.

2. Show that the standard representation and the adjoint representation of $\mathfrak{so}(3)$ are equivalent. Show that the standard representation and the adjoint representation of $\mathrm{SO}(3)$ are equivalent.

3. Determine all invariant subspaces of the standard representation of the complex Heisenberg group.

4. A one-dimensional complex representation of a group G is also called a *character*. Hence, a character of a Lie group G is nothing but a continuous homomorphism $\varphi: G \to \mathbb{C}^\times$.

i) Show that every character of $\mathbb{R}_{>0}$ is of the form $\varphi(x) = |x|^s$ for some $s \in \mathbb{C}$.

ii) Show that every character of $\mathbb{R}^\times$ is of the form $\varphi(x) = \mathrm{sgn}(x)^\varepsilon |x|^s$ for some $s \in \mathbb{C}$ and $\varepsilon \in \{0, 1\}$.1

iii) Show that every character of $\mathrm{GL}(n, \mathbb{R})^+$ is of the form $\varphi(g) = |\det(g)|^s$ for some $s \in \mathbb{C}$. Hint: Use the fact that the derived group of $\mathrm{GL}(n, \mathbb{R})^+$ is $\mathrm{SL}(n, \mathbb{R})$.

iv) Show that every character of $\mathrm{GL}(n, \mathbb{R})$ is of the form $\varphi(g) = \mathrm{sgn}(\det(g))^\varepsilon |\det(g)|^s$ for some $s \in \mathbb{C}$ and $\varepsilon \in \{0, 1\}$.