Math 5303: Lie Groups and Lie Algebras

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Ralf Schmidt

— 6th Assignment —

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22. (Two examples for abstract Lie algebras.)

i) Verify that $\mathbb{R}^3$, with the bracket operation $[x, y] = x \times y$ (classical cross product), becomes a Lie algebra over $\mathbb{R}$.

ii) Let V be a vector space over a field F. Let $\mathsf{gl}(V)$ be the endomorphism ring of V, endowed with the bracket operation $[f, g] = f \circ g - g \circ f$. Verify that $\mathsf{gl}(V)$ is a Lie algebra over F.

23. Let G and H be matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Consider $G \times H$ a matrix Lie group via $(g, h) \mapsto \begin{bmatrix} g & \\ & h \end{bmatrix}$. Show that the Lie algebra of $G \times H$ is isomorphic to the direct sum $\mathfrak{g} \oplus \mathfrak{h}$.

24. Determine a natural basis for the Lie algebra $\mathfrak{sp}(2, \mathbb{R})$ and compute the resulting structure constants. (Arrange the constants in a matrix.)

25. If $\mathfrak{g}$ is a Lie algebra over a field F, then a subalgebra $\mathfrak{h}$ is called an *ideal* if $[X, H] \in \mathfrak{h}$ for all $X \in \mathfrak{g}$ and $H \in \mathfrak{h}$.

i) If $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism, show that the kernel of ϕ is an ideal in $\mathfrak{g}_1$.

ii) The *center* of $\mathfrak{g}$ is the set of all $Z \in \mathfrak{g}$ such that $[X, Z] = 0$ for all $X \in \mathfrak{g}$. Show that the center is an ideal.

iii) The *derived algebra* of $\mathfrak{g}$ is the subspace spanned by all brackets $[X, Y]$, $X, Y \in \mathfrak{g}$. Show that the derived algebra is an ideal.