Math 5303: Lie Groups and Lie Algebras

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— 3rd Assignment —

due Sept. 21, 2007

10. We saw that, given a symmetric, positive matrix $P \in \mathrm{GL}(n, \mathbb{R})$, there exists a symmetric, positive matrix $P^{1/2}$ such that $P^{1/2} \cdot P^{1/2} = P$. Prove that $P^{1/2}$ is unique.

11. Provide a proof of the polar decomposition theorem for $\mathrm{SL}(n, \mathbb{C})$: Given $A \in \mathrm{SL}(n, \mathbb{C})$, there exists a unique pair (U, P) with P self-adjoint and positive, $U \in \mathrm{SU}(n)$, and $A = UP$.

12. For a matrix $X = (X_{i,j}) \in M_n(\mathbb{C})$, let $\|X\| = \left(\sum_{i,j=1}^{n} |X_{i,j}|^2 \right)^{1/2}$. Use standard inequalities from elementary analysis to prove that

i) $\|X + Y\| \leq \|X\| + \|Y\|$

ii) $\|XY\| \leq \|X\| \cdot \|Y\|$

for all $X, Y \in M_n(\mathbb{C})$.

13. Let $V = \{M \in M_2(\mathbb{R}) : \ \mathrm{tr}(M) = 0\}$. The group $\mathrm{GL}(2, \mathbb{R})$ acts on V via

$$(g, X) \longmapsto gXg^{-1} \qquad (g \in \mathrm{GL}(2, \mathbb{R}),\ X \in V).$$

This defines a homomorphism $\mathrm{GL}(2, \mathbb{R}) \to \mathrm{GL}(V)$. If we identify V with $\mathbb{R}^3$ using the basis

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

we obtain a homomorphism $\varphi : \ \mathrm{GL}(2, \mathbb{R}) \to \mathrm{GL}(3, \mathbb{R})$.

i) Compute φ explicitly and show that it is a homomorphism of Lie groups.

ii) Define a bilinear form on V by $\langle X, Y \rangle = \mathrm{tr}(XY)$. Compute the matrix J of this bilinear form with respect to the above basis.

iii) Show that $\mathrm{GL}(2, \mathbb{R})$ preserves the bilinear form, in the sense that $\langle \varphi(g)X, \varphi(g)Y \rangle = \langle X, Y \rangle$ for all $X, Y \in V$ and $g \in \mathrm{GL}(2, \mathbb{R})$. Conclude that the image of φ lies in the special orthogonal group $\mathrm{SO}(J) := \{A \in \mathrm{SL}(3, \mathbb{R}) : \ {}^t\!AJA = J\}$.