Math 5303: Lie Groups and Lie Algebras

Fall 2007, University of Oklahoma

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— 1st Assignment —

due Aug. 29, 2007

1. Let G be a subgroup of $\mathrm{GL}(n,\mathbb{C})$. Show that G is a matrix Lie group if and only if G is closed in $\mathrm{GL}(n,\mathbb{C})$.

2. Consider the orthogonal group $\mathrm{O}(n)=\{A\in\mathrm{GL}(n,\mathbb{R}):\ {}^tAA=1\}$.

i) Show that $\mathrm{O}(n)$ is indeed a group.

ii) Show that $A\in\mathrm{O}(n)\iff {}^tA\in\mathrm{O}(n)$.

iii) For $x,y\in\mathbb{R}^n$ let $\langle x,y\rangle=\sum_{i=1}^n x_iy_i$. Show that $A\in\mathrm{O}(n)$ if and only if

$$\langle Ax,Ay\rangle=\langle x,y\rangle\qquad\text{for all } x,y\in\mathbb{R}^n.$$

3. Describe the groups $\mathrm{SO}(2)$ and $\mathrm{O}(2)$ explicitly. Show that $\mathrm{SO}(2)\cong\mathbb{R}/\mathbb{Z}$.

4. Consider the unitary group $\mathrm{U}(n)=\{A\in\mathrm{GL}(n,\mathbb{C}):\ {}^t\bar{A}A=1\}$.

i) Show that $\mathrm{U}(n)$ is indeed a group.

ii) Show that $A\in\mathrm{U}(n)\iff {}^tA\in\mathrm{U}(n)\iff \bar{A}\in\mathrm{U}(n)$.

iii) For $x,y\in\mathbb{C}^n$ let $\langle x,y\rangle=\sum_{i=1}^n \bar{x}_iy_i$. Show that $A\in\mathrm{U}(n)$ if and only if

$$\langle Ax,Ay\rangle=\langle x,y\rangle\qquad\text{for all } x,y\in\mathbb{C}^n.$$