Math 5303: Lie Groups and Lie Algebras

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— Final Exam —

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Consider the Lie group $G = \mathrm{GL}(3, \mathbb{R})$. Let T be the subgroup of diagonal matrices. Let e_1, e_2, e_3 be the characters (meaning, homomorphisms into $\mathrm{GL}(1, \mathbb{C})$) of T given by

$$e_1(\begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix}) = a, \qquad e_2(\begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix}) = b, \qquad e_3(\begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix}) = c.$$

Even though the characters of T form a multiplicative group, we use additive notation. For example, $2e_1 + e_2 - e_3$ denotes the character

$$\begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} \longmapsto a^2 b c^{-1}.$$

Now consider the action of T on the Lie algebra $\mathfrak{g}$ of G via the adjoint representation. For any non-trivial character α of T of the above form let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \ \mathrm{Ad}(t)X = \alpha(t)X \text{ for all } t \in T\}.$$

Evidently, $\mathfrak{g}_\alpha$ is a subspace of $\mathfrak{g}$. If $\mathfrak{g}_\alpha$ is not zero, then α is called a *root* of G, and $\mathfrak{g}_\alpha$ the corresponding *root space*.

1. Determine all the roots and root spaces for $G = \mathrm{GL}(3, \mathbb{R})$. Express all roots in terms of e_1, e_2, e_3. Interpret e_1, e_2, e_3 as the standard basis vectors of $\mathbb{R}^3$ and draw a coordinate system showing all the roots (this picture is called the *root system*). Show that there exists a subset Δ (a *base*) of the set of roots such that each root can be uniquely expressed as an integral linear combination of the elements of Δ with either only non-negative or only non-positive coefficients. (30 points)

2. Do the same for $G = \mathrm{Sp}(2, \mathbb{R})$ (the root system will live in $\mathbb{R}^2$). (30 points)

3. Do the same for $G = \mathrm{Sp}(3, \mathbb{R})$ (the root system will live in $\mathbb{R}^3$). (40 points)