Dimension Formula for the Spaces of Siegel Cusp Forms of Half Integral Weight and Degree Two

by

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Let $\mathcal{S}_g = \{ Z \in M(g, \mathbb{C}) \mid t^t Z = Z, \ \text{Im } Z > 0 \}$ be the Siegel upper half plane of degree $g$, $\Gamma_g = \text{Sp}(g, \mathbb{Z})$ the Siegel modular group of degree $g$ and

$$\Gamma_g^* = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \mid \text{diagonal elements of } A^t B, C^t D \text{ are even} \right\}.$$

If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we denote $(AZ + B)(CZ + D)^{-1}$ by $M(Z)$. Let $\epsilon(z) = \exp(2\pi i z)$ and for $Z \in \mathcal{S}_g$ put

$$\theta(Z) = \sum_{\eta \in \mathbb{Z}^g} \epsilon\left(\frac{1}{2} \eta^t Z \eta \right).$$

If $M$ belongs to $\Gamma_g^*$, $\theta(M(Z))/\theta(Z)$ is holomorphic on $\mathcal{S}_g$. Let $\alpha = \begin{pmatrix} 2 \cdot 1_g & O \\ O & 1_g \end{pmatrix}$ and let $\theta(Z) = \theta(2Z) = \theta(\alpha(Z))$. Let

$$\Gamma_0^g(4) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \mid C \equiv O \text{ (mod 4)} \right\}.$$

Then $\Gamma_0^g \alpha := \alpha^{-1} \Gamma_0^g \alpha \cap \Gamma_g$ contains $\Gamma_0^g(4)$. Hence if $M$ belongs to $\Gamma_0^g(4)$ or more generally if $M$ belongs to $\Gamma_0^g$, then

$$J(M, Z) := \theta(M(Z))/\theta(Z)$$

is holomorphic on $\mathcal{S}_g$ and satisfies the equality:

$$J(M, Z)^2 = \det(CZ + D)\psi(\det D),$$

where $\psi: 1 + 2\mathbb{Z} \to \{ \pm 1 \}$ is the non-trivial Dirichlet character modulo 4 (cf. §1). $J(M, Z)$ is called the automorphy factor of weight $1/2$.

Let $\mu : GL(g, \mathbb{C}) \to GL(r, \mathbb{C})$ be an irreducible holomorphic representation. $\mu(CZ + D)$ is also an automorphy factor (with respect to $\Gamma_g$) and so is $J(M, Z)^{2k+1} \mu(CZ + D)$ (with respect to $\Gamma_0^g(4)$). Let $\Gamma$ be a subgroup of $\Gamma_0^g(4)$ of finite index. A holomorphic
mapping \( f : \mathcal{S}_g \to \mathbb{C}^r \) is called a Siegel modular form of half integral weight with respect to \( \Gamma \); if \( f \) satisfies the following equality for any \( M \in \Gamma \) and \( Z \in \mathcal{S}_g \):

\[
 f(M \langle Z \rangle) = J(M, Z)^{2k+1} \mu(CZ + D) f(Z).
\]

(We have to assume “the holomorphy at cusps” if \( g = 1 \).) We denote by \( M_{\mu,k+1/2}(\Gamma) \) the \( \mathbb{C} \)-vector space of all such mappings. An element \( f \in M_{\mu,k+1/2}(\Gamma) \) is called a cusp form if \( f \) belongs to the kernels of the \( \Phi \)-operators. We denote the space of cusp forms by \( S_{\mu,k+1/2}(\Gamma) \). Namely, \( f \) belongs to \( S_{\mu,k+1/2}(\Gamma) \) if and only if

\[
 \Phi f(Z_1) := \lim_{\text{Im} Z_2 \to \infty} f(\xi, Z_1, Z_2) = 0
\]

for any \( \xi \in \mathcal{G}_g \) such that \( p(\xi) \in \Gamma_g \), where \( Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \), \( Z_1 \in \mathcal{S}_{g-1} \) and \( Z_2 \in \mathcal{S}_1 \) (cf. Definition 1.5 and Definition 1.7). If \( \mu \) is the trivial representation, we denote \( M_{\mu,k+1/2}(\Gamma) \) and \( S_{\mu,k+1/2}(\Gamma) \) by \( M_{k+1/2}(\Gamma) \) and \( S_{k+1/2}(\Gamma) \), respectively. It is known that \( M_{\mu,k+1/2}(\Gamma) \) is finite-dimensional.

Let \( \chi \) be a character of \( \Gamma \) whose kernel is a subgroup of \( \Gamma \) of finite index. We denote by \( M_{\mu,k+1/2}(\Gamma, \chi) \) the \( \mathbb{C} \)-vector space of the holomorphic mappings of \( \mathcal{S}_g \) to \( \mathbb{C}^r \) which satisfy

\[
 f(M \langle Z \rangle) = J(M, Z)^{2k+1} \chi(M) \mu(CZ + D) f(Z)
\]

for any \( M \in \Gamma \) and \( Z \in \mathcal{S}_g \). We also denote by \( S_{\mu,k+1/2}(\Gamma, \chi) \) its subspace of cusp forms.

Now we assume that \( g = 2 \) and \( \mu \) is the symmetric tensor representation of degree \( j \) which we denote by \( \text{Sym}^j \). We denote \( M_{\mu,k+1/2}(\Gamma) \) and \( S_{\mu,k+1/2}(\Gamma) \) by \( M_{j,k+1/2}(\Gamma) \) and \( S_{j,k+1/2}(\Gamma) \), respectively. Let \( \psi \) be as before. We define a character of \( M \in \Gamma_0^2(4) \) by \( \psi(\det D) \) where \( D \) is the lower right \( 2 \times 2 \) matrix of \( M \). If \( j \) is odd, then \( M_{j,k+1/2}(\Gamma_0^2(4)) \) and \( S_{j,k+1/2}(\Gamma_0^2(4), \psi) \) are \( \{0\} \) since \( -14 \in \Gamma_0^2(4) \) and \( \text{Sym}^j(-12) = -1_{j+1} \). Therefore we assume that \( j \) is even. The purpose of this paper is to compute the dimension of \( S_{2,j,k+1/2}(\Gamma_0^2(4)) \) and \( S_{2,j,k+1/2}(\Gamma_0^2(4), \psi) \) (Theorem 4.4 and Theorem 4.5). From these results we can prove that \( \bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_0^2(4)) \) and \( \bigoplus_{k=0}^{\infty} S_{2,k+1/2}(\Gamma_0^2(4), \psi) \) are free modules of rank one over the graded ring of the automorphic forms of integral weights (Proposition 5.2 and Proposition 5.3). Their structures were explicitly determined by T. Ibukiyama ([Ib]). By using a similar method in [Sto], we can also determine the structure of the module \( \bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_0^2(4)) \) ([T6]).

More generally we can express the dimension of \( S_{j,k+1/2}(\Gamma, \chi) \) by a finite sum for general \( \Gamma \) and \( \chi \) (Theorem 3.2). Especially we will be able to compute the dimension of \( S_{2,j,k+1/2}(\Gamma_0^2(4p), \chi) \), where \( p \) is an odd prime and \( \chi \) is a Dirichlet character modulo \( 4p \) (cf. [T5] for the case of integral weight). But this will be an exhausting job.
1. Transformation formula of $\Theta(Z)$ and the line bundle $\overline{H}_g$

In this section we recall the transformation formula of $\Theta(Z)$ (Theorem 1.4, cf. [Si] or [Smi]). Next we prove that the line bundle of the modular forms of half integral weight is extendable onto the Satake compactification of the Siegel space.

**Definition 1.1.** Let $A \in M(g, C)$ be a symmetric matrix with $\text{Re}(A) > 0$. Then there exists $T \in \text{GL}(g, R)$ such that

$$^tTAT = \begin{pmatrix} 1 + id_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 + id_g \end{pmatrix}.$$  

We define $(\det A)^{1/2} = \det T^{-1} \prod_{j=1}^{g}(1 + id_j)^{1/2}$, where we choose $z^{1/2}$ so that $-\pi/2 < \arg(z^{1/2}) \leq \pi/2$ for $z \in C$.

**Remark 1.2.** If $g = 2$, $(\det A)^{1/2}$ is uniquely determined by the condition $-\pi/2 < \arg(\det A)^{1/2} < \pi/2$, because $-\pi/4 < \arg(1 + id_j)^{1/2} < \pi/4$ ($j = 1, 2$).

**Lemma 1.3.** Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$ and let $m = \text{rank} C$. Then there exist $M', M_1, M_2 \in \Gamma_g$ such that

$$M = M_1 M' M_2, \quad M_1 = \begin{pmatrix} A_1 & B_1 \\ O & D_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} A_2 & O \\ O & D_2 \end{pmatrix},$$

$$M' = \begin{pmatrix} A_0 & O & B_0 & O \\ O & 1_{g-m} & O & O \\ C_0 & O & D_0 & O \\ O & O & O & 1_{g-m} \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} \in \Gamma_m \text{ and } \det C_0 \neq 0.$$  

(If $m = 0$, we suppose $M' = 1_{2g}$.) Moreover we can choose $C_0$ so that

$$C_0 = \begin{pmatrix} c_1 \\ \cdot \\ \cdot \\ c_m \end{pmatrix}, \quad c_i | c_{i+1} \quad (1 \leq i \leq m - 1).$$

**Proof.** The assertion is easily proved ([Smi], Theorem 8.1). But we give a proof here because we use the process of the proof later. There exist $U, V \in GL(g, Z)$ such that $UCV = \begin{pmatrix} C_0 & O \\ O & O \end{pmatrix}$, where $C_0$ has the above form. Let $UD^tV^{-1} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$ ($D_{11} \in M(m, Z)$). Then since $C'D = D^tC$, we have

$$\begin{pmatrix} C_0 & O \\ O & O \end{pmatrix}^t \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}^t \begin{pmatrix} C_0 & O \\ O & O \end{pmatrix}.$$
and \( D_{21} = 0 \). Hence \( D_{21} = 0 \), since \( \det C_0 \neq 0 \). On the other hand
\[
(C_0 \quad O \quad D_{11} \quad D_{12} \\
O \quad O \quad O \quad D_{22})
\]
is primitive. This means that \( D_{22} \in GL(g - m, \mathbb{Z}) \). Let
\[
U_1 = \begin{pmatrix} 1_m & -D_{12}D_{22}^{-1} \\ O & D_{22}^{-1} \end{pmatrix}
\]
and \( D_0 = D_{11} \). Replacing \( U \) with \( U_1U \) we can assume that \( UCV = \begin{pmatrix} C_0 & O \\ O & O \end{pmatrix} \) and
\[
UD^TV^{-1} = \begin{pmatrix} D_0 & O \\ O & 1_{g-m} \end{pmatrix}.
\]
Since \( C_0D_0 = D_0C_0 \), there exists \( M_0 \in \Gamma_m \) such that
\[
M_0 = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}.
\]
We define \( M' \) by using \( M_0 \) as above. Let
\[
M'' = \begin{pmatrix} I_{U^{-1}} & O \\ O & U \end{pmatrix}M \begin{pmatrix} V & O \\ O & I_{V^{-1}} \end{pmatrix}.
\]
Then \( M'M''^{-1} \) has the form
\[
\begin{pmatrix} 1_g & S \\ O & 1_g \end{pmatrix} \quad (S = S). \quad \text{So} \quad M_1 = \begin{pmatrix} I_{U} & -I_US \\ O & U^{-1} \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} V^{-1} & O \\ O & I_V \end{pmatrix}
\]
satisfy the condition.

Now for \( Z \in \mathcal{S}_g \), we put
\[
M'M_2(Z) = \begin{pmatrix} Z_1 & Z_2 \\ I Z_2 & Z_3 \end{pmatrix}, \quad \text{where} \quad Z_1 \in \mathcal{S}_m \quad \text{and} \quad Z_3 \in \mathcal{S}_{g-m}, \quad \text{if} \quad m > 0,
\]
and
\[
j(M, Z) = \begin{cases} 
|\det C_0|^{1/2} \det(-i(Z_1 - A_0C_0^{-1}))^{1/2}, & \text{if} \quad m > 0, \\
1, & \text{if} \quad m = 0.
\end{cases}
\]
Next we put
\[
\lambda(M) = \begin{cases} 
|\det(C_0/2)|^{-1/2} \sum_{\eta \in \mathcal{Z}} \frac{e(-i\eta(C_0^{-1}D_0)\eta)}{e^{-\eta(C_0/2)\eta}}, & \text{if} \quad m > 0, \\
1, & \text{if} \quad m = 0.
\end{cases}
\]
Then we have

**Theorem 1.4.** Let \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^0_0(4) \) and let \( j(M, Z) \) and \( \lambda(M) \) be as above. Let \( J(M, Z) = j(M, Z)^{-1} \lambda(M)^{-1} \). Then it holds that
\[
\Theta(M \langle Z \rangle) = J(M, Z) \Theta(Z)
\]
and
Let $1_g$ be the unit matrix of degree $g$ and $J_g = \begin{pmatrix} O & 1_g \\ -1_g & O \end{pmatrix}$. Let $G_g = \{ M \in GL(2g, \mathbb{R}) \mid M J_g M = v(M) J_g, \text{ with some } v(M) > 0 \}$ be the symplectic group of degree $g$ with similitudes. Let $T = \{ z \in \mathbb{C} \mid |z| = 1 \}$. We define a group $\widetilde{G}_g$ which consists of the pairs $\xi = (M, \phi(Z))$, where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_g$ and $\phi(Z)$ is a non-zero holomorphic function on $\mathcal{S}_g$, such that

$$\phi(Z)^2 = t(\xi) v(M)^{-1/2} \det(CZ + D)$$

for any $Z \in \mathcal{S}_g$ with some $t(\xi) \in T$. The multiplicative law is defined as follows:

$$(M_1, \phi_1(Z))(M_2, \phi_2(Z)) = (M_1 M_2, \phi_1(M_2(Z)) \phi_2(Z)).$$

We denote the natural projection of $\widetilde{G}_g$ to $G_g$ by $p$. By definition, if $p(\xi) = 1_{2g}$, then $\xi = (1_{2g}, t)$ where $t$ is a constant.

**Corollary 1.6.** We have an injective homomorphism $\iota$ of $\Gamma_0^g(4)$ to $\widetilde{G}_g$: $\iota(M) = (M, J(M, Z))$.

**Definition 1.7.** For any holomorphic mapping $f : \mathcal{S}_g \to \mathbb{C}^r$ and $\xi = (M, \phi(Z)) \in \widetilde{G}_g$, we put

$$f \mid [\xi]_{\mu,k+1/2}(Z) = \phi(Z)^{-(2k+1)} \mu(CZ + D)^{-1} f(M(Z)).$$

Then we have

$$f \mid [\xi]_{\mu,k+1/2}(Z) = (f \mid [\xi]_{\mu,k+1/2}) \mid [\eta]_{\mu,k+1/2}(Z)$$

for any $\xi$ and $\eta \in \widetilde{G}_g$. Such a mapping $f$ belongs to $M_{\mu,k+1/2}(\Gamma_0^g(4))$ if and only if $f \mid [\iota(M)]_{\mu,k+1/2}(Z) = f(Z)$ for any $M \in \Gamma_0^g(4)$.

Let $\Gamma_g(N)$ be the principal congruence subgroup of level $N$ of $\Gamma_g$. Namely,

$$\Gamma_g(N) = \{ M \in \Gamma_g \mid M \equiv 1_{2g} \pmod{N} \}.$$
If \( N \geq 3 \), \( V_\mu := \Gamma_1(N)\backslash V_\mu \) is non-singular and is a holomorphic vector bundle over \( \mathcal{X}_g(N) \). \( V_\mu \) is extended to a holomorphic vector bundle \( \tilde{V}_\mu \) on \( \overline{\mathcal{X}}_g(N) \). In the case when \( g = 2 \) and \( \mu = \text{Sym}^l \), we denote \( V_\mu \) and \( \tilde{V}_\mu \) by \( \text{Sym}^l(V) \) and \( \text{Sym}^l(\tilde{V}) \), respectively.

Let \( \mathcal{H}_g \) be \( \mathfrak{S}_g \times \mathbb{C} \). The group \( \Gamma_1(4N) \) acts on \( \mathcal{H}_g \) as follows:

\[
\mathcal{M}(Z, v) = (\mathcal{M}(Z), \mu(CZ + D)v).
\]

Then, \( \mathcal{H}_g := \Gamma_1(4N)\backslash \mathcal{H}_g \) is a holomorphic line bundle over \( \mathcal{X}_g(4N) \). We have

**Theorem 1.8.** The line bundle \( \mathcal{H}_g \) is extendable to an ample line bundle \( \overline{\mathcal{H}}_g \) over the Satake compactification \( \overline{\mathcal{X}}_g(4N) \).

**Proof.** Let \( f \) be a (local) section of \( \mathcal{H}_g^{\otimes (2k+1)} \). Then \( f \) is identified with a (local) modular form of weight \( k + 1/2 \) with respect to \( \Gamma_1(4N) \). We denote \( \phi(Z)^{-(2k+1)} f(P \langle Z \rangle) \) by \( f \mid [\xi]_{k+1/2}(Z) \) for \( \xi = (P, \phi(Z)) \in \tilde{G}_g \). We prove that

\[
f \mid [\xi]_{k+1/2}(Z) = f \mid [\xi]_{k+1/2}(Z + S)
\]

for any \( \xi \in p^{-1}(\Gamma_1) \) and any integral symmetric matrix \( S \) whose entries are divisible by \( 4N \). Then \( f \mid [\xi]_{k+1/2}(Z) \) is expanded to a Fourier series:

\[
f \mid [\xi]_{k+1/2}(Z) = \sum_{T \geq 0} a(T) e(\text{tr}(TZ)/4N),
\]

where \( T \) is over all half-integral semi-positive symmetric matrices and from this fact it is proved that \( \mathcal{H}_g \) is extendable onto \( \overline{\mathcal{X}}_g(4N) \) similarly as in [Sta]. \( \overline{\mathcal{H}}_g^{\otimes 2} \) is isomorphic to the line bundle \( \mathcal{T}_g \) which is defined by the automorphy factor \( \det(CZ + D) \). Since \( \mathcal{T}_g \) is ample ([B]), \( \overline{\mathcal{T}}_g \) is also ample.

Let \( M = \begin{pmatrix} 1_g & S \\ O & 1_g \end{pmatrix} \in \Gamma_1(4N) \) and \( \xi = (P, \phi(Z)) \in p^{-1}(\Gamma_1(4N)) \). Then \( PMP^{-1} \) belongs to \( \Gamma_1(4N) \) since \( \Gamma_1(4N) \) is a normal subgroup of \( \Gamma_1 \). We prove that

\[
\xi \iota(M) \xi^{-1} = \iota(PMP^{-1}).
\]

Then we have

\[
f \mid [\xi \iota(M) \xi^{-1}]_{k+1/2}(Z) = f \mid [\iota(PMP^{-1})]_{k+1/2}(Z) = f(Z)
\]

from the assumption that \( f \) is a (local) modular form with respect to \( \Gamma_1(4N) \). Hence it follows that

\[
f \mid [\xi]_{k+1/2}(Z + S) = f \mid [\xi \iota(M)]_{k+1/2}(Z) = f \mid [\xi]_{k+1/2}(Z).
\]

Now we prove our assertion. Since \( \xi^{-1} = (P^{-1}, \phi(P^{-1}(Z))^{-1}) \), we have

\[
\iota(PMP^{-1}) \xi^{-1} = \iota(PMP^{-1}) \xi \iota(M^{-1}) \xi^{-1} = (1_{2g}, \iota).
\]
where
\[ t = J(P M P^{-1}, P M^{-1} P^{-1} \langle Z \rangle) \phi(M^{-1} P^{-1} \langle Z \rangle) J(M^{-1}, P^{-1} \langle Z \rangle) \phi(P^{-1} \langle Z \rangle)^{-1} \]
is a constant. We prove that \( t = 1 \). Let \( Z = P(Z' + S) \). Since \( J(M^{-1}, P^{-1} \langle Z \rangle) = 1 \), \( t \) is equal to
\[
\frac{\Theta(Z)}{\Theta(P M^{-1} P^{-1} \langle Z \rangle)} \cdot \frac{\phi(M^{-1} P^{-1} \langle Z \rangle)}{\phi(P^{-1} \langle Z \rangle)} \cdot \frac{\Theta(P(Z' + S))}{\Theta(P(Z'))} \cdot \frac{\phi(Z')}{\phi(Z' + S)}.
\]
Let \( P = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \). Then by definition we have
\[
\frac{\phi(Z')}{\phi(Z' + S)} = \sqrt{\det(C Z' + D)} \sqrt{\det(C(Z' + S) + D)}.
\]
Since \( \sqrt{\det(C Z' + D)} \) is a non-zero function on the simply connected space \( S_g \), the sign of \( \sqrt{\det(C(Z' + S) + D)} \) is uniquely determined by the sign of \( \sqrt{\det(C Z' + D)} \) and we have
\[
\lim_{\text{Im} Z' \to \infty} \frac{\phi(Z')}{\phi(Z' + S)} = 1.
\]
Hence the assertion is equivalent to
\[
\lim_{\text{Im} Z' \to \infty} J(P M P^{-1}, P \langle Z' \rangle) = \lim_{\text{Im} Z' \to \infty} \frac{\Theta(P(Z' + S))}{\Theta(P(Z'))} \cdot \frac{\phi(Z')}{\phi(Z' + S)} = 1.
\]
We fix \( P \) and assume that
\[
\lim_{\text{Im} Z' \to \infty} J(P M P^{-1}, P \langle Z \rangle) = 1
\]
for any \( M = \begin{pmatrix} 1_g & S \\ O & 1_g \end{pmatrix} \in \Gamma_g(4N) \). Let \( Q \in \Gamma_{g_0}^g(4) \). Then we have
\[
J(Q P M P^{-1} Q^{-1}, Q P \langle Z \rangle) = J(Q, P M \langle Z \rangle) J(P M P^{-1}, P \langle Z \rangle) J(Q^{-1}, Q P \langle Z \rangle).
\]
Since
\[
\lim_{\text{Im} Z' \to \infty} J(Q, P M \langle Z \rangle) J(Q^{-1}, Q P \langle Z \rangle) = \lim_{\text{Im} Z' \to \infty} \frac{J(Q, P \langle Z + S \rangle)}{J(Q, P \langle Z \rangle)} = 1,
\]
it follows that
\[
\lim_{\text{Im} Z' \to \infty} J(Q P M P^{-1} Q^{-1}, Q P \langle Z \rangle) = 1.
\]
from the assumption.

Next let \( N(B_0, \Gamma_g) \) be the subgroup of \( \Gamma_g \) consisting of the elements of the form:
\[
\begin{pmatrix} U & T^t U^{-1} \\ O & t^t U^{-1} \end{pmatrix}, \quad U \in GL(g, Z), \quad T \in M(g, Z), \quad t^t = T.
\]
Let \( R \in N(B_0, \Gamma_g) \) be an element of the above form. Then
\[ J(P R M R^{-1} P^{-1}, PR(Z)) = J(P M_1 P^{-1}, P(UZ^T U + T)) \]

where

\[ M_1 = \begin{pmatrix} 1_g & U S^T U \\ O & 1_g \end{pmatrix} \]

Hence it follows that

\[ \lim_{\text{Im} Z \to \infty} J(P R M R^{-1} P^{-1}, PR(Z)) = 1 \]

from the assumption.

Therefore it suffices to prove the assertion for the representatives of the double cosets in \( \Gamma_0^g \). Let \( P = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \). Let \( C = (c_{ij}) \) be the matrix such that

\[ C \equiv C \mod 4 \]

and \(-1 \leq c_{ij} \leq 2 \) \((1 \leq i, j \leq g)\). There exists \( P' = \begin{pmatrix} A' & B' \\ C & D' \end{pmatrix} \in \Gamma_g \)

such that \( P \equiv P' \mod 4 \) (cf. [Ig3], Chap. V, Lemma 25). Notice that we can apply the proof of this lemma without changing \( \eta' \) which is the first row of \( C \).

Let \( P' \in \Gamma_g \subset \Gamma_0^g \). Hence we can replace \( P \) with \( P' \). Let \( m = \text{rank} C \) and represent \( P' \) as \( M_1 M' M_2 \) in Lemma 1.3. We can replace \( P' \) with \( M' \).

So we assume that

\[ P = \begin{pmatrix} A_0 & B_0 & O & O \\ O & 1_{g-m} & O & O \\ C_0 & O & D_0 & O \\ O & O & O & 1_{g-m} \end{pmatrix}, \quad \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} \in \Gamma_m, \quad \det C_0 \neq 0 \]

and

\[ C_0 = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}, \quad c_i = 1 \text{ or } 2 \quad (1 \leq i \leq m) \]

It suffices to prove the case when \( N = 1 \). Let \( E_{ij} = (a_{kl}) \) be the matrix such that \( a_{ij} = 1 \) and \( a_{kl} = 0 \), otherwise. Let \( M_1 = \begin{pmatrix} 1_g & S_1 \\ O & 1_g \end{pmatrix} \), \( M_2 = \begin{pmatrix} 1_g & S_2 \\ O & 1_g \end{pmatrix} \in \Gamma_0^g \). Then we have

\[ \lim_{\text{Im} Z \to \infty} J(P M_1 M_2 P^{-1}, P(Z)) \]

\[ = \lim_{\text{Im} Z \to \infty} J(P M_1 P^{-1}, P(Z)) \lim_{\text{Im} Z \to \infty} J(P M_2 P^{-1}, P(Z)) \]

Hence it suffices to prove the assertion for the case when \( S = 4E_{ii} \) or \( S = 4E_{ij} + 4E_{ji} \) \((i \neq j)\). First we prove the case when \( S = 4E_{ii} \). Let \( V_{ij} \) be the matrix corresponding to the transposition \((ij)\). Namely, \( V_{ij} = 1_g - E_{ii} - E_{jj} + E_{ij} + E_{ji} \). Let \( \sigma = (1i) \) and \( V^\sigma = \begin{pmatrix} V_{1i} & O \\ O & V_{ii} \end{pmatrix} \). As we showed before, we have
\[
\lim_{\text{Im } Z \to \infty} J(P M P^{-1}, P \langle Z \rangle) = \lim_{\text{Im } Z \to \infty} J(P^\sigma M^\sigma P^{-1, \sigma}, P\langle V_1 Z V_1 \rangle),
\]
where \( P^\sigma = V^\sigma P V^\sigma \) and \( M^\sigma = V^\sigma M V^\sigma = \left( \begin{array}{cc} 1_g & 4E_{11} \\ O & 1_g \end{array} \right) \). Let \( P^\sigma = \left( \begin{array}{cc} A^\sigma & B^\sigma \\ C^\sigma & D^\sigma \end{array} \right) \).

Then
\[
P^\sigma M^\sigma P^{-1, \sigma} = \left( \begin{array}{cc} 1_g - 4A^\sigma E_{11}^t C^\sigma & 4A^\sigma E_{11}^t A^\sigma \\ -4C^\sigma E_{11}^t C^\sigma & 1_g + 4C^\sigma E_{11}^t A^\sigma \end{array} \right).
\]

If \( i > m \), then the assertion is trivial because \(-4C^\sigma E_{11}^t C^\sigma = O\). So we assume that \( i \leq m \). Then
\[
-4C^\sigma E_{11}^t C^\sigma = \left( \begin{array}{cc} -4c_i^2 & t_o \\ o & O \end{array} \right), \quad 1_g + 4C^\sigma E_{11}^t A^\sigma = \left( \begin{array}{cc} 1 + 4a_i c_i & * \\ o & 1_g - 1 \end{array} \right).
\]

Hence \( P^\sigma M^\sigma P^{-1, \sigma} \) is represented as \( M_1 M' M_2 \) where \( M_2 = 1_{2g} \) and
\[
M' = \left( \begin{array}{cccc}
1 - 4a_i c_i & t_o & 4a_i^2 & t_o \\
0 & 1_g - 1 & o & O \\
-4c_i^2 & t_o & 1 + 4a_i c_i & t_o \\
0 & o & O & 1_g - 1
\end{array} \right).
\]

Let \( P\langle V_1 Z V_1 \rangle = \left( \begin{array}{ccc} W_1 & W_2 & W_3 \\
t W_2 & t W_3 \end{array} \right) \). Then
\[
\lim_{\text{Im } Z \to \infty} W_1 = a_i c_i.
\]

\[
M_0 = \left( \begin{array}{cc} 1 - 4a_i c_i & 4a_i^2 \\ -4c_i^2 & 1 + 4a_i c_i \end{array} \right) \]

fixes \( a_i c_i \). Hence we have
\[
\lim_{\text{Im } Z \to \infty} j(P^\sigma M^\sigma P^{-1, \sigma}, P\langle V_1 Z V_1 \rangle) = \frac{1 - i}{\sqrt{2}}.
\]

On the other hand from Lemma 1.9 exhibited just after this proof we have
\[
\lambda(P^\sigma M^\sigma P^{-1, \sigma}) = \frac{1}{\sqrt{2}} \sum_{x=0}^{2c_i^2-1} e \left( \frac{(1 + 4a_i c_i) x^2}{4c_i^2} \right) = \frac{1 + i}{\sqrt{2}}.
\]

Therefore it follows that
\[
\lim_{\text{Im } Z \to \infty} J(P^\sigma M^\sigma P^{-1, \sigma}, P\langle V_1 Z V_1 \rangle) = 1.
\]
Next we prove the case when \( S = 4E_{ij} + 4E_{ji} \) (\( i \neq j \)). Let \( \sigma = (1i)(2j) \) and \( V^\sigma = \begin{pmatrix} V_{1i}V_{2j} & O \\ O & V_{1i}V_{2j} \end{pmatrix} \). As we showed before, we have
\[
\lim_{\Im Z \to \infty} J(PMP^{-1}, P \langle Z \rangle) = \lim_{\Im Z \to \infty} J(P^\sigma M^\sigma P^{-1}, P^\sigma (V_{1i}V_{2j}Z V_{2j}V_{1i})) .
\]
where \( P^\sigma = V^\sigma PV^\sigma = \begin{pmatrix} A^\sigma & B^\sigma \\ C^\sigma & D^\sigma \end{pmatrix} \) and \( M^\sigma = V^\sigma MV^\sigma = \begin{pmatrix} 1g & 4E_{12} + 4E_{21} \\ O & 1g \end{pmatrix} \).

Then
\[
P^\sigma M^\sigma P^{-1} = \begin{pmatrix} 1g - 4A^\sigma (E_{12} + E_{21})'C^\sigma & 4A^\sigma (E_{12} + E_{21})'A^\sigma \\ -4C^\sigma (E_{12} + E_{21})'C^\sigma & 1g + 4C^\sigma (E_{12} + E_{21})'A^\sigma \end{pmatrix} .
\]
If \( i > m \) or \( j > m \), then the assertion is trivial. So we assume that \( i, j \leq m \). Then
\[
-4C^\sigma (E_{12} + E_{21})'C^\sigma = \begin{pmatrix} 0 & -4c_ic_j \\ -4c_ic_j & 0 \end{pmatrix} ,
\]
\[
1g + 4C^\sigma (E_{12} + E_{21})'A^\sigma = \begin{pmatrix} 1 + 4a_{ij}c_i & 4a_{jj}c_i & 1 \\ 4a_{ii}c_j & 1 + 4a_{jj}c_j & 1 \\ 0 & 0 & 1g-2 \end{pmatrix} ,
\]
\[
1g - 4A^\sigma (E_{12} + E_{21})'C^\sigma = \begin{pmatrix} 1 - 4a_{ij}c_i & -4a_{ii}c_j & 1 \\ -4a_{jj}c_i & 1 - 4a_{jj}c_j & 1 \\ * & * & 1g-2 \end{pmatrix} ,
\]
\[
4A^\sigma (E_{12} + E_{21})'A^\sigma = \begin{pmatrix} 8a_{ii}a_{jj} & 4a_{ii}a_{jj} + 4a_{jj}a_{ij} & 1 \\ 4a_{ii}a_{jj} + 4a_{jj}a_{ij} & 8a_{jj}a_{ii} & 1 \\ * & * & * \end{pmatrix} .
\]

Since \( A^\sigma 'C^\sigma = C^\sigma 'A^\sigma \), we have \( a_{ij}c_i = a_{ji}c_j \). Hence \( P^\sigma M^\sigma P^{-1} \) is represented as \( M_1M_2 \) where \( M_2 = 1g \) and
\[
M' = \begin{pmatrix} 1 - 4a_{ij}c_i & -4a_{ij}c_j & 'o & 8a_{ii}a_{jj} & 4a_{ii}a_{jj} + 4a_{jj}a_{ij} & 'o \\ -4a_{jj}c_i & 1 - 4a_{jj}c_j & 'o & 4a_{ii}a_{jj} + 4a_{ij}a_{jj} & 8a_{jj}a_{ii} & 'o \\ * & * & 1g-2 & 0 & 0 & O \\ 0 & -4c_ic_j & 'o & 1 + 4a_{ij}c_i & 4a_{jj}c_i & 'o \\ -4c_ic_j & 0 & 'o & 4a_{ii}c_j & 1 + 4a_{jj}c_j & 'o \\ 0 & 0 & O & 0 & 0 & 1g-2 \end{pmatrix} .
\]

Let \( P^\sigma (V_{1i}V_{2j}Z V_{2j}V_{1i}) = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} (W_1 \in \mathfrak{S}_2) \). Then
\[
\lim_{\Im Z \to \infty} W_1 = \frac{1}{c_ic_j} \begin{pmatrix} a_{ii}c_i & a_{ij}c_i \\ a_{ji}c_j & a_{jj}c_j \end{pmatrix} .
\]
which is fixed by

\[
M_0 = \begin{pmatrix}
1 - 4a_{ij}c_i & -4a_{ii}c_j & 8a_{ij}a_{ii} & 4a_{ii}a_{jj} + 4a_{ij}a_{ji} \\
-4a_{ij}c_i & 1 - 4a_{jj}c_j & 4a_{ii}a_{jj} + 4a_{ij}a_{ji} & 8a_{jj}a_{ji} \\
0 & -4c_i c_j & 1 + 4a_{ij}a_i & 4a_{jj}c_i \\
-4c_i c_j & 0 & 4a_{ii}c_j & 1 + 4a_{jj}c_j \\
\end{pmatrix}.
\]

Hence we have

\[
\lim_{\Im Z \to \infty} j(P^\sigma M^\sigma P^\sigma^{-1}, P^\sigma (V_{12} V_{22} Z V_{2j} V_{1i})) = 1.
\]

On the other hand from Lemma 1.9 we have

\[
\lambda(P^\sigma M^\sigma P^\sigma^{-1}) = \frac{1}{2c_i c_j} \sum_{x, y=0}^{2c_i c_j-1} e \left( \frac{4a_{ii}c_j x^2 + 2(1 + 4a_{ij}c_i)xy + 4a_{jj}c_i y^2}{4c_i c_j} \right) = 1.
\]

Therefore it follows that

\[
\lim_{\Im Z \to \infty} J(P^\sigma M^\sigma P^\sigma^{-1}, P^\sigma (V_{12} V_{22} Z V_{2j} V_{1i})) = 1.
\]

Now the proof of Theorem 1.8 was completed. \( \square \)

**Lemma 1.9.** (1) If \((c_i, a_{ii}) = (1, 0), (2, 0)\) or \((2, 1)\), then

\[
\sum_{x=0}^{2c_i c_j-1} e \left( \frac{(1 + 4a_{ii}c_i)x^2}{4c_i^2} \right) = (1 + i)c_i.
\]

(2) If \((c_i, c_j, a_{ii}, a_{jj}, a_{ij}, a_{ji}) = (1, 1, 0, 0, 0, 0), (1, 2, 0, 0, 0, 0), (1, 2, 0, 0, 0, 0), (2, 2, 0, 0, 0, 0), (2, 2, 1, 0, 0, 0), (2, 2, 0, 0, 0, 1)\) or \((2, 2, 1, 0, 0, 1)\), then

\[
\sum_{x, y=0}^{2c_i c_j-1} e \left( \frac{4a_{ii}c_j x^2 + 2(1 + 4a_{ij}c_i)xy + 4a_{jj}c_i y^2}{4c_i c_j} \right) = 2c_i c_j.
\]

**Proof.** Directly proved by computation. \( \square \)

**Remark 1.10.** There are some cases such that \( S \) is not divisible by 4, \( P M P^{-1} \in \Gamma_0^g(4) \) and \( \lim_{\Im Z \to \infty} J(P M P^{-1}, P (Z)) = l^s \) \((a \neq 0 \pmod 4)\) (cf. Theorem 3.9 (15) \( \Phi_{15c} \)). Hence \( H_g \) is not extendable onto the Satake compactification \( T^g(\bar{S}_g) \) for general \( \Gamma \).

Actually \( H_g \) is not extendable onto \( T^g_0(4) \backslash \bar{S}_g \).

**Notation 1.11.** Let \( \bar{H}_g \) and \( \bar{T}_g \) be as above. Then we denote by \( \bar{H}_g \) and \( \tilde{L}_g \) the pullbacks of \( \bar{H}_g \) and \( \bar{T}_g \) by the natural morphism of \( \bar{X}_g(4N) \) to \( \bar{X}_g(4N) \), respectively.

### 2. Classification of the fixed points (sets)

Let \( \Gamma \) be a subgroup of \( \Gamma_0^g(4) \) of finite index. If \( g \geq 2 \), \( \Gamma \) contains \( \Gamma_g(4N) \) for some \( N \) ([BLS], [Me]). In the following we assume that \( g = 2 \) and \( \mu \) is Sym\(^2\). The space of
Siegel modular forms $M_{j,k+1/2}(\Gamma_2(4N))$ is canonically identified with the space

$$\Gamma(\tilde{X}_2(4N), \mathcal{O}(\text{Sym}^j(\tilde{V}) \otimes \tilde{H}_2^{(2k+1)})).$$

which is the space of the global holomorphic sections of $\text{Sym}^j(\tilde{V}) \otimes \tilde{H}_2^{(2k+1)}$. The divisor at infinity $D := \tilde{X}_2(4N) - X_2(4N)$ is a divisor with simple normal crossings. The space of cusp forms $S_{j,k+1/2}(\Gamma_2(4N))$ is canonically identified with the space

$$\Gamma(\tilde{X}_2(4N), \mathcal{O}(\text{Sym}^j(\tilde{V}) \otimes \tilde{H}_2^{(2k+1)} - D)).$$

Here $\mathcal{O}(\text{Sym}^j(\tilde{V}) \otimes \tilde{H}_2^{(2k+1)} - D)$ is the sheaf of germs of holomorphic sections which vanish along $D$ and this is isomorphic to $\mathcal{O}(\text{Sym}^j(\tilde{V}) \otimes \tilde{H}_2^{(2k+1)} \otimes [D]^{(1)}$ where $[D]$ is the holomorphic line bundle which is associated with $D$.

Let $\chi$ be a character of $\Gamma$ whose kernel is a subgroup of $\Gamma$ of finite index. We may assume that the kernel of $\chi$ contains $\Gamma_2(4N)$. Let $f \in S_{j,k+1/2}(\Gamma_2(4N))$ and $M \in \Gamma$. We define an action of $M$ on $S_{j,k+1/2}(\Gamma_2(4N))$ as follows:

$$Mf(M(Z)) = J(M, Z)^{2k+1} \chi(M) \text{Sym}^j(CZ + D)f(Z).$$

Since $\Gamma_2(4N)$ acts trivially on $S_{j,k+1/2}(\Gamma_2(4N))$, this action induces an action of the factor group $\Gamma/\Gamma_2(4N)$ on $S_{j,k+1/2}(\Gamma_2(4N))$ and $S_{j,k+1/2}(\Gamma, \chi)$ is identified with the invariant subspace of $S_{j,k+1/2}(\Gamma_2(4N))$. Thus we have

$$S_{j,k+1/2}(\Gamma, \chi) = S_{j,k+1/2}(\Gamma_2(4N))^{\Gamma/\Gamma_2(4N)}.$$  

Therefore the dimension of $S_{j,k+1/2}(\Gamma, \chi)$ is computed by applying the holomorphic Lefschetz fixed point formula ([AS]) and the vanishing theorem (Theorem 4.1) to the above situation.

To use the holomorphic Lefschetz fixed point formula we have to classify the fixed points (sets) of $\Gamma_2$ and $\Gamma_2/\Gamma_2(4N)$ acting on $\tilde{X}_2(4N)$. We classify the (irreducible components of) the fixed points (sets) of $\Gamma_2$ in the following sense. Let $\Phi$ and $\Phi'$ be the fixed points (sets). $\Phi$ and $\Phi'$ are called equivalent if there is an element of $\Gamma_2$ which maps $\Phi$ biholomorphically to $\Phi'$. The fixed points in the quotient space $X_2(4N)$ were classified in [G]. The fixed points in the divisor at infinity are classified easily. In total there are 25 kinds of fixed points (sets).

**Lemma 2.1.** Among the 25 kinds of fixed points (sets) the following 10 fixed points (sets) are not fixed by the elements of $\Gamma_2$ which are conjugate to elements of $\Gamma_0^2(4)$, where $\rho = e(1/3)$, $\omega = e(1/5)$, $\eta = (1 + 2\sqrt{-2})/3$ and $Z \in \mathbb{S}_1$. To represent the fixed points (sets) we use the same notations $\Phi_7, \Phi_8, \ldots, \Phi_{21}$ as in [T2].
\[ \Phi_7 : \left\{ \begin{pmatrix} i & 0 \\ 0 & Z \end{pmatrix} \right\}, \quad \Phi_8 : \left\{ \begin{pmatrix} \rho & 0 \\ 0 & i \end{pmatrix} \right\}, \quad \Phi_9 : \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \Phi_{11} : \begin{pmatrix} \rho & 0 \\ 0 & i \end{pmatrix}, \quad \Phi_{13} : \begin{pmatrix} \eta & 0 \\ \eta - 1/2 & \eta \end{pmatrix}, \quad \Phi_{14} : \begin{pmatrix} \omega + \omega^3 & \omega - \omega^4 \\ \omega + \omega^3 & -\omega \end{pmatrix}, \quad \Phi_{18} : \begin{pmatrix} i & 0 \\ 0 & \infty \end{pmatrix}, \quad \Phi_{19} : \begin{pmatrix} i & 0 \\ \infty & \infty \end{pmatrix}, \quad \Phi_{20} : \begin{pmatrix} \rho & 0 \\ 0 & \infty \end{pmatrix}, \quad \Phi_{21} : \begin{pmatrix} \rho & (\rho + 2)/3 \\ \infty & \infty \end{pmatrix}. \]

**Proof.** If \( M \) belongs to \( \Gamma_0^2(4) \), we have
\[
M \equiv \begin{pmatrix} U & V \\ O & U^{-1} \end{pmatrix} \pmod{4},
\]
where \( U \in GL(2, \mathbb{Z}) \). Since \((\det U)^2 \equiv 1 \) and \((x \cdot 12 - U^{-1}) \equiv \det(x \cdot 12 - U^{-1}) \cdot (\det U)^2 \equiv \det(x U - 12) \cdot \det U \pmod{4} \), the characteristic polynomial \( P_M(x) \) of \( M \) is equivalent to one of the following polynomials modulo 4:

\[
\begin{aligned}
(x^2 + 1)^2, \\
(x^2 + x + 1)^2, \\
(x^2 + x - 1)(x^2 - x - 1), \\
(x^2 + 2x + 1)^2, \\
(x^2 - 1)^2, \\
(x^2 - x + 1)^2, \\
(x^2 - x - 1)(x^2 + x - 1), \\
(x^2 + 2x - 1)(x^2 - 2x - 1).
\end{aligned}
\]

Therefore if \( M \in \Gamma_2 \) is conjugate to an element of \( \Gamma_0^2(4) \), then the characteristic polynomial \( P_M(x) \) of \( M \) is equivalent to one of the following three polynomials modulo 4:

\[
x^4 + 2x^2 + 1, \\
x^4 + 2x^3 + 3x^2 + 2x + 1, \\
x^4 + x^2 + 1.
\]

From this fact we can show that the above points (sets) except \( \Phi_9 \) are not fixed by the elements of \( \Gamma_2 \) which are conjugate to elements of \( \Gamma_0^2(4) \). Since the characteristic polynomial of \( P_2 \) (cf. Proposition 2.5) which fixes \( \Phi_9 \) is \((x^2 + 1)^2\), the above argument is not valid in this case. In this case we have to check more carefully and the assertion is proved in Theorem 2.8 (9).

**Remark 2.2.** Although we represented \( \Phi_7 \) by \( \left\{ \begin{pmatrix} i & 0 \\ 0 & Z \end{pmatrix} \right\} \subset \mathfrak{S}_2 \) symbolically, \( \Phi_7 \) means the image of \( \left\{ \begin{pmatrix} i & 0 \\ 0 & Z \end{pmatrix} \right\} \) to \( \tilde{X}_2(4N) \). The same applies to \( \Phi_8 \) and also to the following cases.

The remaining 15 fixed points (sets) have the contributions to the dimension formula. But since the automorphic factor \( J(M, Z) \) is defined with respect to \( \Gamma_0^2(4) \), we have to classify the remaining 15 fixed points (sets) with respect to \( \Gamma_0^2(4) \). Let \( \Phi \) be one of the following 15 fixed points (sets):
We call Φ an infinite group, it is not an easy task to classify Γ, because the computation many times in the following.

Let (Z₁ Z₁₂) ∈ 𝒮₂. Z₁, Z₁₂ ∈ 𝒮₁ and W ∈ ℂ. Strictly speaking Φ₁₇ should be represented as

\[
\left\{ \begin{array}{c} \left( Z \begin{array}{cc} 1/2 & \infty \\ 1/2 & \infty \end{array} \right) \\
\left( Z \begin{array}{cc} 2N + 1/2 & \infty \\ 2N + 1/2 & \infty \end{array} \right) \\
\left( Z \begin{array}{cc} 2NZ + 1/2 & \infty \\ 2NZ + 1/2 & \infty \end{array} \right) \\
\left( Z \begin{array}{cc} 2N(Z + 1) + 1/2 & \infty \\ 2N(Z + 1) + 1/2 & \infty \end{array} \right) \end{array} \right\}.
\]

This appears as a boundary of Φ₃ and is a four fold cover of a one-dimensional cusp.

**Definition 2.3.** Let us denote by Fix(M) the fixed points in ̃X₂(4N) of M and let

\[
C(Φ) = \{ M \in Γ₂ | Φ is an irreducible component of Fix(M) \},
\]

\[
C^p(Φ) = \{ M \in C(Φ) | Φ is an irreducible component of Fix(M) \},
\]

\[
C(Φ, Γ₂) = \{ M \in Γ₂ | M(Z) = Z for any Z ∈ Φ \},
\]

\[
C^p(Φ, Γ₂) = \{ M \in C(Φ, Γ₂) | Φ is an irreducible component of Fix(M) \},
\]

\[
N(Φ, Γ₂) = \{ M \in Γ₂ | M maps Φ into Φ \}.
\]

We call C^p(Φ) and C^p(Φ, Γ₂) the sets of proper elements in C(Φ) and in C(Φ, Γ₂), respectively.

What we have to do is to classify the double cosets in Γ₀^a(4) \ Γ₂ | N(Φ, Γ₂). Since Γ₂ is an infinite group, it is not an easy task to classify Γ₀^a(4) \ Γ₂ | N(Φ, Γ₂). But since Γ₀^a(4) contains Γ₂(4) which is a normal subgroup of Γ₂, we can take the quotient by Γ₂(4) and reduce the problem to a task in the finite group Γ₂/Γ₂(4) ≃ Sp(2, ℤ/Z) and we can use a computer. So first we classify Γ₀^a(4) \ Γ₂ which consists of 120 cosets and next classify these cosets with respect to the action of N(Φ, Γ₂) from the right. We have to execute this computation many times in the following.

Let P₁, P₂, · · · , Pₙ be the representatives of Γ₀^a(4) \ Γ₂ | N(Φ, Γ₂). Next we have to check Pᵢ(Γ₀^a(4) \ Γ₂) Pᵢ⁻¹ ∩ Γ₀^a(4) (i = 1, 2, · · · , n) is empty or not. Let
$$P_i \Phi = \{ P_i (Z) \mid Z \in \Phi \}.$$  

The following assertion is trivial.

**Lemma 2.4.** If $P_i C^P(\Phi, \Gamma_2) P_i^{-1} \cap \Gamma_0^2(4)$ is empty, then $P_i \Phi$ is not fixed by the elements of $\Gamma_0^2(4)$.

Before we classify the fixed points (sets), we classify the rational boundary components of $\Sigma_2$ with respect to $\Gamma_0^2(4)$ and determine the configuration of the cusps of the Satake compactification $\Gamma_0^2(4) \backslash \Sigma_2$. Let $B_1$ be the one-dimensional boundary component of $\Sigma_2$ which is defined by $\text{Im} Z_2 = \infty$. Let $N(B_1, \Gamma_2)$ be the stabilizer in $\Gamma_2$ of $B_1$. The elements of $N(B_1, \Gamma_2)$ have the following form:

$$\begin{pmatrix}
* & 0 & * & * \\
* & * & * & * \\
* & 0 & * & * \\
0 & 0 & 0 & *
\end{pmatrix}.$$  

The one-dimensional cusps of the Satake compactification correspond bijectively to the double cosets in $\Gamma_2 \backslash N(B_1, \Gamma_2)$. Similarly as above we classify the double cosets by a computer. We have

**Proposition 2.5.** $\Gamma_0^2(4) \backslash \Gamma_2 / N(B_1, \Gamma_2)$ consists of four double cosets. The representatives are $P_1, P_2, P_3$ and $P_4$, where $P_1 = I_4$ and

$$P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}.$$  

Let

$$M = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  

The submatrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $M$ acts on the one-dimensional rational boundary component at infinity and $P_i M P_i^{-1}$ $(i = 1, 2, 3, 4)$ belongs to $\Gamma_0^2(4)$ if and only if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

belongs to $\Gamma_0^4(4)$, respectively. Hence each one-dimensional cusps of the Satake compactification is biholomorphic to $\Gamma_0^4(4) \backslash \Sigma_1$. $\Gamma_0^4(4) \backslash \Sigma_1$ is a rational curve with three holes.

Let $B_0$ be the zero-dimensional boundary component of $\Sigma_2$ which is defined by $\text{Im} Z_1 = \text{Im} Z_2 = \infty$. Let $N(B_0, \Gamma_2)$ be the stabilizer in $\Gamma_2$ of $B_0$. The elements of
\( N(B_0, \Gamma_2) \) have the following form:
\[
\begin{pmatrix}
* & * & * & * \\
* & * & * & * \\
0 & 0 & * & * \\
0 & 0 & * & * \\
\end{pmatrix}.
\]

The zero-dimensional cusps of the Satake compactification correspond bijectively to the double cosets in \( \Gamma_0^2(4) \backslash \Gamma_2/N(B_0, \Gamma_2) \). We have

**Proposition 2.6.** \( \Gamma_0^2(4) \backslash \Gamma_2/N(B_0, \Gamma_2) \) consists of seven double cosets. The representatives are \( P_1, P_2, \ldots, P_7 \), where

\[
P_5 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
\end{pmatrix}, \quad P_6 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 2 & 0 & 1 \\
1 & 0 & 0 & 0 \\
\end{pmatrix}, \quad P_7 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
2 & 0 & 1 & 0 \\
0 & 2 & 0 & 1 \\
\end{pmatrix}.
\]

Let \( C_i \) be the one-dimensional cusp corresponding to the double coset \( \Gamma_0^2(4) P_i N(B_1, \Gamma_2) \) (\( i = 1, 2, 3, 4 \)), respectively and let \( Q_i \) be the zero-dimensional cusp corresponding to the double coset \( \Gamma_0^2(4) P_i N(B_0, \Gamma_2) \) (\( i = 1, 2, \ldots, 7 \)), respectively. Then the cusps of the Satake compactification look like as follows.

This is proved as follows. The cusps \( Q_1, Q_2, Q_3 \) and \( Q_4 \) are on \( C_1, C_2, C_3 \) and \( C_4 \), respectively. Since

\[
\Gamma_0^2(4) P_3 N(B_1, \Gamma_2) = \Gamma_0^2(4) P_1 N(B_1, \Gamma_2),
\]
\[
\Gamma_0^2(4) P_6 N(B_1, \Gamma_2) = \Gamma_0^2(4) P_3 N(B_1, \Gamma_2),
\]
\[
\Gamma_0^2(4) P_7 N(B_1, \Gamma_2) = \Gamma_0^2(4) P_4 N(B_1, \Gamma_2),
\]
\[
Q_5, Q_6 \text{ and } Q_7 \text{ are on } C_1, C_3 \text{ and } C_3, \text{ respectively. Let } P_{11} \text{ be as in Theorem 2.8. Since}
\]
\[
\Gamma_0^2(4) P_{11} N(B_0, \Gamma_2) = \Gamma_0^2(4) P_{11} N(B_0, \Gamma_2),
\]
\[
\Gamma_0^2(4) P_{11} N(B_1, \Gamma_2) = \Gamma_0^2(4) P_{11} N(B_1, \Gamma_2),
\]
$Q_5$ is also on $C_4$. Similarly we can prove that $Q_5$ and $Q_6$ are also on $C_2$, $Q_5$ is also on $C_1$ and $Q_7$ is also on $C_4$.

**Proposition 2.7.** We have $[\Gamma^g_2 : \Gamma^g_0(4)] = 2^{g(g-1)/2}$. Especially $[\Gamma^g_2 : \Gamma^g_0(4)] = 2$ and $\Gamma^g_0(4)$ is a normal subgroup of $\Gamma^g_2$.

**Proof.** We have

$$\alpha \Gamma^g_0 \alpha^{-1} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^g_0 \mid B \equiv O \pmod{2}, \text{ diagonal elements of } C' D \text{ are even} \right\},$$

$$\alpha \Gamma^g_0(4) \alpha^{-1} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^g_0 \mid B \equiv C \equiv 0 \pmod{2} \right\}.$$

We map them into $Sp(g, F_2)$. Namely,

$$\alpha \Gamma^g_0 \alpha^{-1} / \Gamma^g_0(2) = \left\{ \begin{pmatrix} A & O \\ TA & A^{-1} \end{pmatrix} \mid A \in GL(g, F_2), T = T, \right\}$$

diagonal elements of $T$ are 0.

$$\alpha \Gamma^g_0(4) \alpha^{-1} / \Gamma^g_0(2) = \left\{ \begin{pmatrix} A & O \\ O & A^{-1} \end{pmatrix} \mid A \in GL(g, F_2) \right\}.$$

Hence $[\alpha \Gamma^g_0 \alpha^{-1} : \Gamma^g_0(2)] = 2^{g(g-1)/2} |GL(g, F_2)|$ and $[\alpha \Gamma^g_0(4) \alpha^{-1} : \Gamma^g_0(2)] = |GL(g, F_2)|$. Therefore $[\Gamma^g_0 : \Gamma^g_0(4)] = [\alpha \Gamma^g_0 \alpha^{-1} : \alpha \Gamma^g_0(4) \alpha^{-1}] = 2^{g(g-1)/2}$. \hfill \Box

As a matter of fact we classify the fixed points (sets) with respect to $\Gamma^g_2$ instead of $\Gamma^g_0(4)$ (cf. Remark 3.6). In the following theorem we represent the representatives with respect to $\Gamma^g_2$ instead of $\Gamma^g_0(4)$ as $\Phi_a, \Phi_a', \Phi_b, \Phi_c$. These notations mean that $\Phi_a$ and $\Phi_a'$ are equivalent with respect to $\Gamma^g_2$ and $\Phi_b$ and $\Phi_c$ are not equivalent with respect to $\Gamma^g_2$.

**Theorem 2.8.** Let

\[
\begin{align*}
P_8 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, & P_9 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{pmatrix}, & P_{10} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix}, \\
P_{11} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ -1 & 2 & 0 & 0 \end{pmatrix}, & P_{12} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{pmatrix}, & P_{13} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \end{pmatrix}, \\
P_{14} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix}, & P_{15} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, & P_{16} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix}. 
\end{align*}
\]
Then fixed points (sets) of $\Gamma_0^2(4)$ are classified as follows.

1. $\Gamma_0^2(4) \backslash \Gamma_2 \cap N(\Phi_1, \Phi_2)$ consists of one double coset. The representative is $P_1$. $\Phi_{1a} := P_1 \Phi_1$ is the total space $\tilde{X}_{23}(4N)$.

2. $\Gamma_0^2(4) \backslash \Gamma_2 \cap N(\Phi_2, \Phi_2)$ consists of three double cosets. The representatives are $P_1$, $P_4$ and $P_8$. Only $\Phi_{2a} := P_1 \Phi_2$ and $\Phi_{2a'} := P_4 \Phi_2$ are fixed by elements of $\Gamma_0^2(4)$.

3. $\Gamma_0^2(4) \backslash \Gamma_2 \cap N(\Phi_3, \Phi_2)$ consists of five double cosets. The representatives are $P_1$, $P_2$, $P_5$, $P_6$ and $P_7$. Only $\Phi_{3a} := P_1 \Phi_3$, $\Phi_{3b} := P_5 \Phi_3$ and $\Phi_{3c} := P_7 \Phi_3$ are fixed by elements of $\Gamma_0^2(4)$.

4. $\Gamma_0^2(4) \backslash \Gamma_2 \cap N(\Phi_4, \Phi_2)$ consists of eleven double cosets. The representatives are $P_1$, $P_3$, $P_4$, $P_5$, $P_6$, $P_9$, $P_{10}$, $P_{11}$, $P_{12}$, $P_{13}$ and $P_{14}$. Only $\Phi_{4a} := P_1 \Phi_4$ and $\Phi_{4a'} := P_4 \Phi_4$ are fixed by elements of $\Gamma_0^2(4)$.
(5) $\Gamma_0^2(4) \backslash \Gamma_2 / N(\Phi_5, \Gamma_2)$ consists of eight double cosets. The representatives are $P_1, P_2, P_6, P_7, P_{10}, P_{13}, P_{15}$ and $P_{16}$. Only $\Phi_{5a} := P_5 \Phi_5$ and $\Phi_{5b} := P_5 \Phi_5$ are fixed by elements of $\Gamma_0^2(4)$.

(6) $\Gamma_0^2(4) \backslash \Gamma_2 / N(\Phi_6, \Gamma_2)$ consists of six double cosets. The representatives are $P_1, P_3, P_5, P_9, P_{14}$ and $P_{16}$. Only $\Phi_{6a} := P_6 \Phi_6$ is fixed by elements of $\Gamma_0^2(4)$.

(9) $\Phi_9$ is not fixed by the elements of $\Gamma_2$ which are conjugate to elements of $\Gamma_0^2(4)$.

(10) $\Gamma_0^2(4) \backslash \Gamma_2 / N(\Phi_{10}, \Gamma_2)$ consists of ten double cosets. The representatives are $P_1, P_3, P_4, P_7, P_9, P_{10}, P_{13}, P_{15}, P_{18}, P_{19}, P_{20}, P_{21}, P_{22}, P_{23}, P_{24}, P_{27}, P_{28}, P_{29}, P_{30}, P_{31}, P_{32}$ and $P_{33}$. Only $\Phi_{10a} := P_{14} \Phi_{10}$ is fixed by elements of $\Gamma_0^2(4)$.

(12) $\Gamma_0^2(4) \backslash \Gamma_2 / N(\Phi_{12}, \Gamma_2)$ consists of twenty four double cosets. The representatives are $P_1, P_3, P_4, P_7, P_9, P_{10}, P_{13}, P_{15}, P_{18}, P_{19}, P_{20}, P_{21}, P_{22}, P_{23}, P_{24}, P_{27}, P_{28}, P_{29}, P_{30}, P_{31}, P_{32}$ and $P_{33}$. Only $\Phi_{12a} := P_{24} \Phi_{12}$ and $\Phi_{12a'} := P_{29} \Phi_{12}$ are fixed by elements of $\Gamma_0^2(4)$.

(15) $\Gamma_0^2(4) \backslash \Gamma_2 / N(\Phi_{15}, \Gamma_2)$ consists of four double cosets. The representatives are $P_1, P_2, P_3$ and $P_4$. Let $\Phi_{15a} := P_1 \Phi_{15}$, $\Phi_{15a'} := P_4 \Phi_{15}$, $\Phi_{15b} := P_2 \Phi_{15}$ and $\Phi_{15c} := P_3 \Phi_{15}$. All of them are fixed by elements of $\Gamma_0^2(4)$.

(16) $\Gamma_0^2(4) \backslash \Gamma_2 / N(\Phi_{16}, \Gamma_2)$ consists of seven double cosets. The representatives are $P_1, P_2, P_3, P_4, P_5, P_7$ and $P_{12}$. Only $\Phi_{16a} := P_1 \Phi_{16}$, $\Phi_{16a'} := P_4 \Phi_{16}$, $\Phi_{16b} := P_2 \Phi_{16}$, $\Phi_{16c} := P_3 \Phi_{16}$, $\Phi_{16d} := P_{12} \Phi_{16}$ and $\Phi_{16e} := P_{34} \Phi_{16}$ are fixed by elements of $\Gamma_0^2(4)$.

(17) $\Gamma_0^2(4) \backslash \Gamma_2 / N(\Phi_{17}, \Gamma_2)$ consists of ten double cosets. The representatives are $P_1, P_3, P_4, P_7, P_9, P_{11}, P_{13}$ and $P_{14}$. Only $\Phi_{17a} := P_1 \Phi_{17}$, $\Phi_{17a'} := P_4 \Phi_{17}$, $\Phi_{17b} := P_3 \Phi_{17}$, $\Phi_{17c} := P_5 \Phi_{17}$, $\Phi_{17c'} := P_{11} \Phi_{17}$, $\Phi_{17d} := P_7 \Phi_{17}$ and $\Phi_{17e} := P_{14} \Phi_{17}$ are fixed by elements of $\Gamma_0^2(4)$.

(22) $\Gamma_0^2(4) \backslash \Gamma_2 / N(\Phi_{22}, \Gamma_2)$ consists of twelve double cosets. The representatives are $P_1, P_2, P_3, P_4, P_5, P_7, P_{11}, P_{12}, P_{13}, P_{17}$ and $P_{32}$. Let $\Phi_{22a} := P_1 \Phi_{22}$, $\Phi_{22a'} := P_4 \Phi_{22}$, $\Phi_{22b} := P_2 \Phi_{22}$, $\Phi_{22b'} := P_7 \Phi_{22}$, $\Phi_{22c} := P_{12} \Phi_{22}$, $\Phi_{22d} := P_{13} \Phi_{22}$, $\Phi_{22e} := P_{32} \Phi_{22}$, $\Phi_{22f} := P_3 \Phi_{22}$, $\Phi_{22f'} := P_{11} \Phi_{22}$, $\Phi_{22g} := P_5 \Phi_{22}$, $\Phi_{22h} := P_7 \Phi_{22}$ and $\Phi_{22h'} := P_{12} \Phi_{22}$. All of them are fixed by elements of $\Gamma_0^2(4)$.

(23) $\Gamma_0^2(4) \backslash \Gamma_2 / N(\Phi_{23}, \Gamma_2)$ consists of fifteen double cosets. The representatives are $P_1, P_2, P_3, P_4, P_7, P_{11}, P_{12}, P_{13}, P_{17}, P_{31}, P_{32}$ and $P_{34}$. Only $\Phi_{23a} := P_1 \Phi_{23}$, $\Phi_{23b} := P_2 \Phi_{23}$, $\Phi_{23b'} := P_7 \Phi_{23}$, $\Phi_{23c} := P_3 \Phi_{23}$, $\Phi_{23c'} := P_{11} \Phi_{23}$, $\Phi_{23d} := P_4 \Phi_{23}$, $\Phi_{23d'} := P_{12} \Phi_{23}$, $\Phi_{23e} := P_5 \Phi_{23}$, $\Phi_{23e'} := P_{13} \Phi_{23}$, $\Phi_{23f} := P_{31} \Phi_{23}$, $\Phi_{23f'} := P_7 \Phi_{23}$, $\Phi_{23g} := P_{17} \Phi_{23}$ and $\Phi_{23g'} := P_{32} \Phi_{23}$ are fixed by elements of $\Gamma_0^2(4)$.

(24) $\Gamma_0^2(4) \backslash \Gamma_2 / N(\Phi_{24}, \Gamma_2)$ consists of thirteen double cosets. The representatives are $P_1, P_2, P_3, P_4, P_7, P_{11}, P_{12}, P_{13}, P_{17}, P_{32}$ and $P_{34}$. Only $\Phi_{24a} := P_1 \Phi_{24}$, $\Phi_{24a'} := P_4 \Phi_{24}$, $\Phi_{24b} := P_2 \Phi_{24}$, $\Phi_{24b'} := P_{12} \Phi_{24}$, $\Phi_{24c} := P_3 \Phi_{24}$, $\Phi_{24c'} := P_{11} \Phi_{24}$, $\Phi_{24d} := P_5 \Phi_{24}$, $\Phi_{24d'} := P_{17} \Phi_{24}$, and $\Phi_{24e} := P_{32} \Phi_{24}$ are fixed by elements of $\Gamma_0^2(4)$.  

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\( \Gamma_0^2(4) \backslash \Gamma_2 / N(\Phi_{25}, \Gamma_2) \) consists of eight double cosets. The representatives are \( P_1, P_2, P_3, P_4, P_5, P_6, P_7 \) and \( P_{11} \).

Let \( \Phi_{25a} := P_1 \Phi_{25}, \Phi_{25a'} := P_4 \Phi_{25}, \Phi_{25b} := P_5 \Phi_{25}, \Phi_{25c} := P_3 \Phi_{25}, \Phi_{25d} := P_8 \Phi_{25}, \Phi_{25d'} := P_{11} \Phi_{25}, \Phi_{25e} := P_6 \Phi_{25} \) and \( \Phi_{25f} := P_7 \Phi_{25} \). All of them are fixed by elements of \( \Gamma_0^2(4) \).

**Proof.** We prove only (9). Other cases are similarly proved. \( C^p(\Phi_0, \Gamma_2) \) has ten elements. It consists of \( \pm P_2, \pm P_3, \pm P_5, \pm P_7 \) and other four elements. Other four elements are conjugate to \( \pm P_3 \) or \( \pm P_5^{-1} \). Since the characteristic polynomials of \( \pm P_3 \) and \( \pm P_5^{-1} \) are \( x^4 + 1 \), they are not conjugate to elements of \( \Gamma_0^2(4) \) (cf. Proof of Lemma 2.1). \( \Gamma_0^2(4) \backslash \Gamma_2 / N(\Phi_9, \Gamma_2) \) consists of eighteen double cosets. The representatives are \( P_1, P_3, P_4, P_7, P_9, P_{10}, P_{12}, P_{17}, P_{18}, P_{19}, P_{20}, P_{21}, P_{22}, P_{23}, P_{24}, P_{25} \) and \( P_{26} \). \( P_i P_i P_i^{-1} \) (\( i = 1, 3, 4, 7, 8, 9, 10, 12, 17, \ldots, 26 \)) does not belong to \( \Gamma_0^2(4) \). Hence \( \Phi_0 \) is not fixed by the elements of \( \Gamma_2 \) which are conjugate to elements of \( \Gamma_0^2(4) \) (cf. Lemma 2.4). \( \square \)

3. **Detailed data**

In this section we list the data which we use to compute the dimension formula. First we recall the holomorphic Lefschetz fixed point formula. Let \( X \) be a compact complex manifold and \( V \) a holomorphic vector bundle on \( X \), and let \( G \) be a finite group of automorphisms of the pair \((X, V)\). For \( g \in G \) let \( X^g \) be the set of fixed points of \( g \). Then, \( X^g \) is a disjoint union of submanifolds of \( X \). Let

\[
X^g = \sum_{a} X^g_a
\]

be the irreducible decomposition of \( X^g \), and let

\[
N^g_a = \sum_{\theta} N^g_a(\theta)
\]

denote the normal bundle of \( X^g_a \) decomposed according to the eigenvalues \( e^{i\theta} \) of \( g \). We put

\[
U^\theta(N^g_a(\theta)) = \prod_{\beta} \left( \frac{1 - e^{-x_{\beta} - i\theta}}{1 - e^{-i\theta}} \right)^{-1},
\]

where the Chern class of \( N^g_a(\theta) \) is

\[
c(N^g_a(\theta)) = \prod_{\beta}(1 + x_{\beta}).
\]

Let \( T(X^g_a) \) be the Todd class of \( X^g_a \). Let \( V|X^g_a \) be the restriction of \( V \) to \( X^g_a \) and \( ch(V|X^g_a)(g) \) the Chern character of \( V|X^g_a \) with \( g \)-action ([AS]). Put

\[
\tau(g, X^g_a) = \frac{ch(V|X^g_a)(g) \cdot \prod_{\theta} U^\theta(N^g_a(\theta)) \cdot T(X^g_a)}{\det(1 - g(N^g_a)^*)}[X^g_a]
\]

\( G \) acts on \( \Gamma_0^2 \). For each \( \lambda \in \Gamma_0^2 \) we have the character \( \chi_{\lambda} = \frac{1}{|\lambda|} \sum_{\theta} \lambda^{i\theta} \). Then, \( \chi_{\lambda} \) is a character of \( G \). For each \( \lambda \in \Gamma_0^2 \) we have

\[
\chi_{\lambda}(g) = \frac{1}{|\lambda|} \sum_{\theta} e^{i\theta} \lambda(g)^{i\theta} = \prod_{\beta} (1 + x_{\beta})^{|\lambda|}.
\]

Let \( \Phi_{25} \) be the set of fixed points of \( \Phi_{25} \). Let \( \Phi_9 \) be the set of fixed points of \( \Phi_9 \). Then, \( \Phi_{25} \) is a disjoint union of submanifolds of \( \Phi_9 \). Let \( \Phi_{25} = \bigcup_{\theta} \Phi_{25}(\theta) \) be the decomposition of \( \Phi_{25} \) according to the eigenvalues \( e^{i\theta} \) of \( g \).

\[
\Phi_{25}(\theta) = \prod_{\beta} \left( \frac{1 - e^{-x_{\beta} - i\theta}}{1 - e^{-i\theta}} \right)^{-1},
\]

where the Chern class of \( \Phi_{25}(\theta) \) is

\[
c(\Phi_{25}(\theta)) = \prod_{\beta}(1 + x_{\beta}).
\]

Let \( T(\Phi_{25}) \) be the Todd class of \( \Phi_{25} \). Let \( V|\Phi_{25} \) be the restriction of \( V \) to \( \Phi_{25} \) and \( ch(V|\Phi_{25})(g) \) the Chern character of \( V|\Phi_{25} \) with \( g \)-action ([AS]). Put

\[
\tau(g, \Phi_{25}) = \frac{ch(V|\Phi_{25})(g) \cdot \prod_{\theta} U^\theta(\Phi_{25}(\theta)) \cdot T(\Phi_{25})}{\det(1 - g(\Phi_{25})^*)}[\Phi_{25}]
\]
and
\[ \tau(g) = \sum_a \tau(g, X_a^2). \]

We have

**Theorem 3.1 ([AS]).**
\[ \sum_{i \geq 0} (-1)^i \text{Tr}(g|H^i(X, \mathcal{O}(V))) = \tau(g). \]

Let \( \Gamma \) be a subgroup of \( \Gamma_0^2(4) \) of finite index and \( \chi \) a character of \( \Gamma \) whose kernel is a subgroup of \( \Gamma \) of finite index. The kernel of \( \chi \) contains \( \Gamma_2(4N) \) for some \( N \). In our case \( X, V \) and \( G \) are \( \tilde{X}_2(4N), \text{Sym}^j(\tilde{V}) \otimes \tilde{H}_2^{\otimes(k+1)} \otimes [D]^{\otimes(-1)} \) and \( \Gamma/\Gamma_2(4N) \), respectively. But in the following we assume that \( V \) is \( \text{Sym}^j(\tilde{V}) \otimes \tilde{H}_2^{\otimes(k)} \otimes [D]^{\otimes(-1)} \) for the sake of simplicity. When we apply the data, we replace \( k \) with \( 2k + 1 \).

Applying the holomorphic Lefschetz theorem we have the dimension formula. We state the general dimension formula (cf. [T5], Theorem 1.6). Let \( \Gamma \) be a subgroup of \( \Gamma_0^2(4) \). We denote the centralizer of \( g \) in \( \Gamma_0^2(4)/\Gamma_2(4N) \) and in \( \Gamma/\Gamma_2(4N) \) by \( C(g, \Gamma_0^2(4)/\Gamma_2(4N)) \) and \( C(g, \Gamma/\Gamma_2(4N)) \), respectively. Let
\[ N(\Phi, \Gamma_0^2(4)/\Gamma_2(4N)) = \{ M \in \Gamma_0^2(4)/\Gamma_2(4N) \mid M \text{ maps } \Phi \text{ into } \Phi \}. \]

**Theorem 3.2.** Under the assumption that the higher cohomology groups vanish, the dimension of \( S_{j,k+1/2}(\Gamma, \chi) \) is expressed as
\[ \sum_{\Phi} \sum_P \sum_{M} \frac{\tau(PM P^{-1}, P\Phi)}{|N(P\Phi, \Gamma_0^2(4)/\Gamma_2(4N))|} \left( \sum_{g} \frac{|C(g, \Gamma_0^2(4)/\Gamma_2(4N))|}{|C(g, \Gamma/\Gamma_2(4N))|} \chi(g) \right). \]

Here \( \Phi \) is over the 15 fixed points (sets) in \( \mathcal{S}_2 \), \( P \) is over the representatives of \( \Gamma_0^2(4)/\Gamma_2(4N) \) and \( M \) is over \( C^p(\Phi) \cap P^{-1} \Gamma_0^2(4) P \). Let \( \text{Conj}(\Gamma/\Gamma_2(4N)) \) be the set of the representatives of the conjugacy classes of \( \Gamma/\Gamma_2(4N) \). Moreover \( g \) runs over \( \text{Conj}(\Gamma/\Gamma_2(4N)) \) such that \( g \) is conjugate to \( PM P^{-1} \) in \( \Gamma_0^2(4)/\Gamma_2(4N) \).

Let \( \Phi \) be an irreducible component of fixed points sets and let \( M \in C^p(\Phi) \). The Chern character with \( M \)-action \( ch : W \mapsto ch(W)(M) \) is also a ring homomorphism of the ring of the holomorphic vector bundles to the cohomology ring as in the case of the usual Chern character. Hence we have
\[ ch(V|\Phi)(M) = ch(\text{Sym}^j(\tilde{V}) \otimes \tilde{H}_2^{\otimes(k)} \otimes [D]^{\otimes(-1)}|\Phi)(M) \]
\[ = ch(\text{Sym}^j(\tilde{V})|\Phi)(M) \otimes ch(\tilde{H}_2^{\otimes(k)}|\Phi)(M) \otimes ch([D]^{\otimes(-1)}|\Phi)(M). \]

Let \( Z \in \Phi \). Then by definition we have
\[ ch(\tilde{H}_2^{\otimes(k)}|\Phi)(M) = J(M, Z)^k ch(\tilde{H}_2^{\otimes(k)}|\Phi). \]

Let
\[ ch_0(V|\Phi)(M) = ch(\text{Sym}^j(\tilde{V})|\Phi)(M) \otimes ch(\tilde{H}_2^{\otimes(k)}|\Phi) \otimes ch([D]^{\otimes(-1)}|\Phi)(M) \]
and let
\[ \tau_0(M, \Phi) = \left\{ \frac{\text{ch}(V|\Phi)(M) \cdot \prod_{\theta} U^\theta(NM(\theta)) \cdot T(\Phi)}{\det(1 - M(NM)^*)} \right\} \Phi. \]

Then we have
\[ \text{ch}(V|\Phi)(M) = J(M, Z)^4 \text{ch}(V|\Phi)(M) \]
and
\[ \tau(M, \Phi) = J(M, Z)^k \tau_0(M, \Phi). \]

Let \( \bar{L}_2 \) and \( \bar{L}_2 \) be as in Notation 1.11. We have \( \bar{H}_2^{\otimes 2} \simeq \bar{L}_2 \) and
\[
\text{ch}(\bar{H}_2) = 1 + c_1(\bar{H}_2) + \frac{1}{2} c_1(\bar{H}_2)^2 + \frac{1}{6} c_1(\bar{H}_2)^3
\]
\[
= 1 + \frac{1}{2} c_1(\bar{L}_2) + \frac{1}{8} c_1(\bar{L}_2)^2 + \frac{1}{48} c_1(\bar{L}_2)^3.
\]

Since \( \text{Sym}^i(V) \) and \( \bar{L}_2 \) correspond to the automorphy factors (which are defined with respect to \( \Gamma_2 \)) \( \text{Sym}^i(CZ + D) \) and \( \det(CZ + D) \), respectively and the divisor \( D \) is invariant with respect to \( \Gamma_2 \), the terms in \( \tau_0(M, \Phi) \) are invariant with respect to \( \Gamma_2 \). Namely, we have

**Proposition 3.3.** Let \( M \in C^\ell(\Phi, \Gamma_2) \) and \( P \in \Gamma_2 \). If \( M \) and \( PMP^{-1} \) belong to \( \Gamma_0^2(4) \), then
\[
\tau_0(PMP^{-1}, P \Phi) = \tau_0(M, \Phi).
\]

Hence the only term in \( \tau(M, \Phi) \) which depends on \( \Gamma_0^2(4) \) is \( J(M, Z) \). What we have to do to get the dimension formula is to compute \( |N(P \Phi, \Gamma_0^2(4)/\Gamma_2(4N))| \) and \( \tau(PMP^{-1}, P \Phi) \) for every \( P \in \Gamma_0^2(4), \Gamma_2/N(\Phi, \Gamma_2) \) and \( M \in C^\ell(\Phi) \cap P^{-1} \Gamma_0^2(4) \).

From the above observation it suffices to compute \( \tau_0(M, \Phi) \) and \( |N(P \Phi, \Gamma_0^2(4)/\Gamma_2(4N))| \) and \( J(PMP^{-1}, P \langle Z \rangle) \) \((Z \in \Phi)\). We list \( \tau_0(M, \Phi) \) in Theorem 3.4, \( |N(P \Phi, \Gamma_0^2(4)/\Gamma_2(4N))| \) in Theorem 3.8 and \( J(PMP^{-1}, P \langle Z \rangle) \) \((Z \in \Phi)\) in Theorem 3.9, respectively.

In the following theorem we assume that \( j \) is even. Hence we replace \( j \) with \( 2j \) and assume \( G \) is \( \Gamma_0^2(4)/\pm \Gamma_2(4N) \). The notations \( \varphi_1, \varphi_2, \cdots, \varphi_{25}(6, r, s, t) \in \Gamma_2/\pm \Gamma_2(4N) \) are same as in \([T2]\). We do not show them explicitly here. If one does not know them, he can obtain the dimension formula from the data in Theorem 3.4, Theorem 3.8 and Theorem 3.9. The elements in \( C^\ell(\Phi_{10}) \) except \( \varphi_{10}(i) \) \((i = 1, 2, 4, 5)\) are not conjugate to the elements in \( \Gamma_0^2(4) \).

**Theorem 3.4.** Let \( V \) be \( \text{Sym}^{2j}(\bar{V}) \otimes \bar{H}_2^{\otimes k} \otimes |D|^{\otimes (-1)} \). Let \( \xi = e(1/4N) \) and \( \rho = e(1/3) \). We have the following results. There \( p \) in \( \prod \) is over the odd prime numbers which divide \( N \), while \( \text{Tr}_\rho \) means the trace map \( Q(\rho) \rightarrow Q \).

\[
\tau_0(\varphi_1, \Phi_1) = 2^{33} s^{-1}(2j + 1)(2(k - 4)(4j + k - 2)(2j + k - 3)N^{10}
- 30(2j + k - 3)N^8 + 45N^7) \prod (1 - p^{-2})(1 - p^{-4})
\]
(2) \( \tau_0(\varphi_2, \Phi_2) = 2^{-1}( (k - 4)(4j + k - 2)N^6 - 6(2j + k - 3)N^5 + 36N^4)\prod(1 - p^{-2})^2 \)

(3) \( \tau_0(\varphi_3, \Phi_3) = 2^{-2}( (k - 4)(4j + k - 2)N^6 - 3(2j + k - 3)N^5 + 3N^4)\prod(1 - p^{-2})^2 \)

(4) \( \tau_0(\varphi_4, \Phi_4) = 2^{-1}( (2j + k - 3)N^3 - 3N^2)\prod(1 - p^{-2}) \)

(5) \( \tau_0(\varphi_5, \Phi_5) = 2^{-1}3^{-1}( (2j + k - 3)N^3 - 2N^2)\prod(1 - p^{-2}) \)

(6) \( \tau_0(\varphi_6, \Phi_6) = \text{Tr}_p(\rho^j(1 - \rho))((2j + 2k - 3)N^3 - 9N^2) \)

\[
\tau_0(\varphi_6^{-1}, \Phi_6) = \tau_0(\varphi_6, \Phi_6)
\]

(10) \( \tau_0(\varphi_{10}(1), \Phi_{10}) = 3^{-2}(\rho)^j(2\rho + 1)(2j + 1) \)

(11) \( \tau_0(\varphi_{10}(2), \Phi_{10}) = 3^{-2}(\rho^2)^j(2\rho^2 + 1)(2j + 1) \)

(12) \( \tau_0(\varphi_{12}, \Phi_{12}) = 2^{-1}3^{-1}\text{Tr}_p((\rho)^j(-\rho^2)) \)

(15) \( \tau_0(\varphi_{15}(r), \Phi_{15}) = 2^{-3}3^{-1}(2j + 1)N^3\prod(1 - p^{-2}) \)

\[
\times \left( \frac{9 - (2j + 2k - 3)N}{(1 - \zeta^r)} + \frac{2j + 2k - 3N - 6}{(1 - \zeta^r)^2} - \frac{4}{(1 - \zeta^r)^3} \right)
\]

(16) \( \tau_0(\varphi_{16}(r), \Phi_{16}) = 2^{-5}3^{-1}\left( \frac{12 - (2j + 2k - 3)N}{(1 - \zeta^r)} \right)N^3\prod(1 - p^{-2}) \)

(17) \( \tau_0(\varphi_{17}(r), \Phi_{17}) = \left( \frac{8 - (2j + 2k - 3)N}{(1 - \zeta^r)} + \frac{4}{(1 - \zeta^r)^2} \right)N^3\prod(1 - p^{-2}) \)

(22) \( \tau_0(\varphi_{22}(1, r, t), \Phi_{22}) = \frac{(2j + 1)}{(\zeta^r - 1)(\zeta^t - 1)} \left( \frac{2}{(\zeta^r - 1)} + \frac{2}{(\zeta^t - 1)} + 3 \right) \)

(23) \( \tau_0(\varphi_{23}(2, r, t), \Phi_{23}) = 2^{-1}(-1)^j(\zeta^{r+t} - 1)^{-1} \)

(24) \( \tau_0(\varphi_{24}(2, r, t), \Phi_{24}) = 2^{-1}(-1)^j(\zeta^{r+t} - 1)^{-1} \)
\[
(25) \quad \tau_0(\varphi_2(1, r, s, t), \Phi_{25}) = (2j + 1)(\zeta^{t+s} - 1)^{-1}(\zeta^{t+s} - 1)^{-1}(\zeta^t - 1)^{-1}
\]
\[
\tau_0(\varphi_2(2, r, s, t), \Phi_{25}) = 3^{-1}\text{Tr}_\rho(\rho^j(1 - \rho))(\zeta^{t+s+r} - 1)^{-1}
\]
\[
\tau_0(\varphi_2(3, r, s, t), \Phi_{25}) = 3^{-1}\text{Tr}_\rho(\rho^j(1 - \rho))(\zeta^{t+s+r} - 1)^{-1}
\]
\[
\tau_0(\varphi_2(4, r, s, t), \Phi_{25}) = (\zeta^{t+2s+r} - 1)^{-1}(\zeta^s - 1)^{-1}
\]
\[
\tau_0(\varphi_2(5, r, s, t), \Phi_{25}) = (\zeta^{s+t} - 1)^{-1}(\zeta^t - 1)^{-1}
\]
\[
\tau_0(\varphi_2(6, r, s, t), \Phi_{25}) = (\zeta^{t+s} - 1)^{-1}(\zeta^t - 1)^{-1}
\]

**Proof.** Due to [T5], Theorem 3.2 which is the result in the case of weight \(k\) and level \(N\). It suffices to remove \(\det(CZ + D)^k\) of \(\tau(\varphi, \Phi)\) in [T5], Theorem 3.2 and replace \(k\) and \(N\) with \(k/2\) and \(4N\), respectively. \(\square\)

Let \(\Gamma_2^N\) be as above. We have the following

**PROPOSITION 3.5.** If \(\Gamma_2^N \triangleright \Phi, \Gamma_2 = \Gamma_2^N \triangleright \Phi, \Gamma_2\), then
\[
|N(\Phi, \Gamma_2^N(4)/\Gamma_2(4N))| = |N(\Phi, \Gamma_2^N(4)/\Gamma_2(4N))|
\]

**Proof.** From the assumption we have elements \(\gamma \in \Gamma_2^N\) and \(n \in N(\Phi, \Gamma_2)\) such that \(P' = \gamma P\). \(N(\Phi, \Gamma_2^N(4)/\Gamma_2(4N))\) is isomorphic to \((N(\Phi, \Gamma_2) \cap \Gamma_2^N(4))/N(\Phi, \Gamma_2) \cap \Gamma_2(4N))\). Since \(\Gamma_2^N(4)\) is a normal subgroup of \(\Gamma_2^N\) and \(\Gamma_2(4N)\) is a normal subgroup of \(\Gamma_2\), we have
\[
\gamma (N(\Phi, \Gamma_2) \cap \Gamma_2^N(4)) \gamma^{-1} = N(\Phi, \Gamma_2) \cap \Gamma_2^N(4).
\]
\[
\gamma (N(\Phi, \Gamma_2) \cap \Gamma_2(4N)) \gamma^{-1} = N(\Phi, \Gamma_2) \cap \Gamma_2(4N).
\]

The assertion is proved from these relations. \(\square\)

Let
\[
C(P, \Phi, \Gamma_2^N(4)/\Gamma_2(4N)) = \{M \in \Gamma_2^N(4)/\Gamma_2(4N) \mid M(Z) = Z \text{ for any } Z \in P \Phi\}
\]
and let \(C^p(P, \Phi, \Gamma_2^N(4)/\Gamma_2(4N))\) be the set of proper elements.

**REMARK 3.6.** Let \(P \Phi, P' \Phi\) and \(\gamma\) be as in the above proposition. It is obvious that \(C^p(P, \Phi, \Gamma_2^N(4)/\Gamma_2(4N)) = \gamma C^p(P, \Phi, \Gamma_2^N(4)/\Gamma_2(4N)) \gamma^{-1}\). Since the automorphy factor \(J(M, Z)\) is defined with respect to \(\Gamma_2^N\), we have
\[
J(\gamma M \gamma^{-1}, \gamma Z) = J(M, Z)
\]
for \(M \in C^p(P, \Phi, \Gamma_2^N(4)/\Gamma_2(4N))\) and \(Z \in P \Phi\). From the above proposition and this observation it follows that the contributions of \(P \Phi\) and \(P' \Phi\) to the dimension of \(S_{2,j,k+1/2}(\Gamma_2^N(4), \psi)\) are same.

Next we prove that the contributions of \(P \Phi\) and \(P' \Phi\) are the same also in the case of \(S_{2,j,k+1/2}(\Gamma_2^N(4), \psi)\). It suffices to prove the following
Lemma 3.7. Let $M = \begin{pmatrix} A & B \\ 4C & D \end{pmatrix} \in \Gamma_0^2(4)$ and $\gamma \in \Gamma_2^\infty$. Put $\tilde{\psi}(M) = \psi(\det D)$. Then $\tilde{\psi}(M) = \tilde{\psi}(\gamma M \gamma^{-1})$.

Proof. It suffices to prove that $\tilde{\psi}$ is extendable to a character of $\Gamma_2^\infty$. Let $P_4$ be as in Proposition 2.5. Then $\Gamma_2^\infty = \Gamma_0^2(4) \cup \Gamma_0^2(4) P_4$. Let $M = \begin{pmatrix} A & B \\ 4C & D \end{pmatrix}$, $M' = \begin{pmatrix} A' & B' \\ 4C' & D' \end{pmatrix} \in \Gamma_0^2(4)$ and $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$MP_4 = \begin{pmatrix} A + 2BF & B \\ 4C + 2DF & D \end{pmatrix}.$$ 

Put $\tilde{\psi}(MP_4) = \psi(\det D)$. We have to prove that $\tilde{\psi}(MM'P_4) = \tilde{\psi}(M)\tilde{\psi}(M'P_4)$, $\tilde{\psi}(MP_4M') = \tilde{\psi}(MP_4)\tilde{\psi}(M')$ and $\tilde{\psi}(MP_4M'P_4) = \tilde{\psi}(MP_4)\tilde{\psi}(M'P_4)$ for any $M, M' \in \Gamma_0^2(4)$. The first case is trivial. We prove only the second case. The third case is similarly proved. The lower right $2 \times 2$ matrix of $MP_4M'$ is $(4C + 2DF)B' + DD'$. Let

$$D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}, B' = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \text{ and } D' = \begin{pmatrix} d_{11}' & d_{12}' \\ d_{21}' & d_{22}' \end{pmatrix}.$$ 

Then

$$\det((4C + 2DF)B' + DD') = (\det D')(\det D') \equiv 2(d_{11}d_{22} + d_{12}d_{21}) \mod 4.$$ 

On the other hand we have

$$b_{11}'d_{12}' + b_{21}'d_{22}' = b_{12}'d_{11}' + b_{22}'d_{21}' \mod 4,$$

because it holds that $B'B' \equiv B'D'$. Hence it follows that

$$\det((4C + 2DF)B' + DD') \equiv (\det D')(\det D') \mod 4.$$ 

This proves the assertion. \hfill $\Box$

In the following theorem we list $|N(P \Phi, \Gamma_0^2(4)/\Gamma_2(4N))|$. If $P \Phi$ and $P' \Phi$ are not equivalent with respect to $\Gamma_0^2(4)$ but equivalent with respect to $\Gamma_2^\infty$, we list only one of them and we mark the notations of the fixed points (sets) by *. We also list the order of $C(P \Phi, \Gamma_0^2(4)/\Gamma_2(4N))$. We list $P \Phi, |C(P \Phi, \Gamma_0^2(4)/\Gamma_2(4N))|$ and $|N(P \Phi, \Gamma_0^2(4)/\Gamma_2(4N))|$ in this order. Similarly as before $p$ in $\prod$ is over the odd prime numbers which divide $N$.

**Theorem 3.8.** The orders of the isotropy groups and the stabilizer groups of the fixed points (sets) of $\Gamma_0^2(4)$ are as follows.

| $\Phi$ | $\prod$ | $|N(P \Phi, \Gamma_0^2(4)/\Gamma_2(4N))|$ |
|-------|-------|------------------------------------------|
| $\Phi_{1a}$ | $2 \cdot 3N \prod(1 - p^{-2})(1 - p^{-4})$ |
| $\Phi_{2a}*$ | $4 \cdot 7N^6 \prod(1 - p^{-2})^2$ |
| $\Phi_{3a}$ | $4 \cdot 10N^6 \prod(1 - p^{-2})^2$ |
\[ \Phi_{3b} \quad 4 \quad 2^8 N^6 \prod(1 - p^{-2})^2 \]
\[ \Phi_{3c} \quad 4 \quad 2^{10} N^6 \prod(1 - p^{-2})^2 \]

(4) \[ \Phi_{4a} \quad 8 \quad 2^5 N^3 \prod(1 - p^{-2}) \]
(5) \[ \Phi_{5a} \quad 8 \quad 2^5 N^3 \prod(1 - p^{-2}) \]
\[ \Phi_{5b} \quad 8 \quad 2^5 N^3 \prod(1 - p^{-2}) \]

(6) \[ \Phi_{6a} \quad 12 \quad \begin{cases} 2^3 N^3 \prod(1 - p^{-2}), & \text{if } 3 \nmid N \\ 2^3 3 N^3 \prod(1 - p^{-2}), & \text{if } 3 \mid N \end{cases} \]
(10) \[ \Phi_{10a} \quad 12 \quad 12 \]
(12) \[ \Phi_{12a} \quad 24 \quad 24 \]

(15) \[ \Phi_{15a} \quad 8N \quad 2^{10} N^6 \prod(1 - p^{-2}) \]
\[ \Phi_{15b} \quad 2N \quad 2^6 N^6 \prod(1 - p^{-2}) \]
\[ \Phi_{15c} \quad 8N \quad 2^9 N^6 \prod(1 - p^{-2}) \]

(16) \[ \Phi_{16a} \quad 16N \quad 2^6 N^4 \prod(1 - p^{-2}) \]
\[ \Phi_{16b} \quad 4N \quad 2^4 N^4 \prod(1 - p^{-2}) \]
\[ \Phi_{16c} \quad 16N \quad 2^6 N^4 \prod(1 - p^{-2}) \]
\[ \Phi_{16d} \quad 16N \quad 2^6 N^4 \prod(1 - p^{-2}) \]
\[ \Phi_{16e} \quad 4N \quad 2^4 N^4 \prod(1 - p^{-2}) \]

(17) \[ \Phi_{17a} \quad 16N \quad 2^8 N^4 \prod(1 - p^{-2}) \]
\[ \Phi_{17b} \quad 16N \quad 2^8 N^4 \prod(1 - p^{-2}) \]
\[ \Phi_{17c} \quad 16N \quad 2^7 N^4 \prod(1 - p^{-2}) \]
\[ \Phi_{17d} \quad 16N \quad 2^8 N^4 \prod(1 - p^{-2}) \]
\[ \Phi_{17e} \quad 4N \quad 2^5 N^4 \prod(1 - p^{-2}) \]

(22) \[ \Phi_{22a} \quad 2^5 N^2 \quad 2^9 N^3 \]
\[ \Phi_{22b} \quad 2^5 N^2 \quad 2^3 N^3 \]
\[ \Phi_{22c} \quad 2^5 N^2 \quad 2^9 N^3 \]
\[ \Phi_{22d} \quad 2^4 N^2 \quad 2^6 N^3 \]
\( \Phi_{22e} \)  \( 2^3N^2 \)  \( 2^6N^3 \)  
\( \Phi_{22f} \)  \( 2^3N^2 \)  \( 2^6N^3 \)  
\( \Phi_{22g} \)  \( 2^3N^2 \)  \( 2^5N^3 \)  
\( \Phi_{22h} \)  \( 2^5N^2 \)  \( 2^8N^3 \)  

(23)  
\( \Phi_{23a} \)  \( 2^7N^2 \)  \( 2^7N^2 \)  
\( \Phi_{23b} \)  \( 2^3N^2 \)  \( 2^3N^2 \)  
\( \Phi_{23c} \)  \( 2^6N^2 \)  \( 2^6N^2 \)  
\( \Phi_{23d} \)  \( 2^4N^2 \)  \( 2^4N^2 \)  
\( \Phi_{23e} \)  \( 2^4N^2 \)  \( 2^4N^2 \)  
\( \Phi_{23f} \)  \( 2^7N^2 \)  \( 2^7N^2 \)  
\( \Phi_{23g} \)  \( 2^4N^2 \)  \( 2^4N^2 \)  

(24)  
\( \Phi_{24a} \)  \( 2^7N^2 \)  \( 2^7N^2 \)  
\( \Phi_{24b} \)  \( 2^6N^2 \)  \( 2^6N^2 \)  
\( \Phi_{24c} \)  \( 2^4N^2 \)  \( 2^4N^2 \)  
\( \Phi_{24d} \)  \( 2^7N^2 \)  \( 2^7N^2 \)  
\( \Phi_{24e} \)  \( 2^4N^2 \)  \( 2^4N^2 \)  

(25)  
\( \Phi_{25a} \)  \( 2^83N^3 \)  \( 2^83N^3 \)  
\( \Phi_{25b} \)  \( 2^23N^3 \)  \( 2^23N^3 \)  
\( \Phi_{25c} \)  \( 2^8N^3 \)  \( 2^8N^3 \)  
\( \Phi_{25d} \)  \( 2^6N^3 \)  \( 2^6N^3 \)  
\( \Phi_{25e} \)  \( 2^5N^3 \)  \( 2^5N^3 \)  
\( \Phi_{25f} \)  \( 2^8N^3 \)  \( 2^8N^3 \)  

**Proof.** We prove only the cases of \( \Phi_{ab} = P_5\Phi_3 \) and \( \Phi_{3c} = P_7\Phi_3 \). Other cases are proved easily. \( N(P\Phi_3, \Gamma_2^2(4))/\Gamma_2(4N) \) is isomorphic to \( (N(P\Phi_3, \Gamma_2) \cap \Gamma_2^2(4))/\Gamma_2(4N) \). From [T2], Theorem 2.2 we have

\[
[N(P\Phi_3, \Gamma_2) : N(P\Phi_3, \Gamma_2) \cap \Gamma_2(4N)] = [PN(\Phi_3, \Gamma_2) \cap \Gamma_2(4N)]
\]

\[
= 2^{113}N^6 \prod (1 - p^{-2}).
\]
So it suffices to determine \([N(P\Phi_3, \Gamma_2) : N(P\Phi_3, \Gamma_2) \cap \Gamma_0^2(4)]\). Let \(\varepsilon, \delta\) and \(\gamma\) be

\[
\begin{pmatrix}
1 & 0 & 0 & 1/2 \\
0 & 1 & 1/2 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\text{ and }
\begin{pmatrix}
0 & 0 & 1/2 & 0 \\
0 & 0 & 0 & 1/2 \\
-2 & 0 & 0 & 0 \\
0 & -2 & 0 & 0
\end{pmatrix},
\]

respectively. Let

\[
N_1 = \begin{pmatrix}
a_1 & 0 & b_1/2 & 0 \\
0 & a_2 & 0 & b_2/2 \\
2c_1 & 0 & d_1 & 0 \\
0 & 2c_2 & 0 & d_2
\end{pmatrix} \in Sp(2, \mathbb{Q})
\begin{pmatrix}
a_i & b_i & c_i & d_i \\
\varepsilon & \delta & \varepsilon & d_i \mod 2
\end{pmatrix}
\]

Then \([N(\Phi_3, \Gamma_2) = \varepsilon N_1 \varepsilon^{-1} \cup \delta \varepsilon N_1 \varepsilon^{-1}\) ([T2], Theorem 2.6). Let \(l\) be a natural number and let \(N_1(2l)\) be the subgroup of \(N_1\) consisting of the elements such that

\[
\begin{pmatrix}
a_i & b_i \\
c_i & d_i
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \pmod{2l} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \end{pmatrix} \quad (i = 1, 2). \quad N_1(2l) \text{ is a normal subgroup of } N_1 \text{ which is isomorphic to } \Gamma_1(2l) \times \Gamma_1(2l) \text{ and we have } [N_1 : N_1(2)] = 6.
\]

Let \(N_2\) be the subgroup of \(N_1\) consisting of the elements such that \(c_2 \equiv 0 \pmod{4}\) and \(b_1 \equiv c_2 \pmod{8}\) Then

\[
P_5 N(\Phi_3, \Gamma_2) P_5^{-1} \cap \Gamma_0^2(4) = P_5(\varepsilon N_2 \varepsilon^{-1}) P_5^{-1} \cup P_5(\delta \varepsilon \gamma N_2 \varepsilon^{-1}) P_5^{-1}
\]

and

\[
P_5 N(\Phi_3, \Gamma_2) P_5^{-1} = P_5 N(\Phi_3, \Gamma_2) P_5^{-1} \cap \Gamma_0^2(4) = [N_1 : N_2].
\]

On the other hand

\[
[N_1 : N_1(8)] = 6 \cdot [N_1(2) : N_1(8)] = 6 \cdot [\Gamma_1(2) : \Gamma_1(8)] = 3 \cdot 2^{13}
\]

and \([N_2 : N_1(8)] = 2^{10}\) because \(N_2/N_1(8)\) is isomorphic to a subgroup of \(SL(2, \mathbb{Z}/8\mathbb{Z}) \times SL(2, \mathbb{Z}/8\mathbb{Z})\) of order \(2^{10}\). Hence we have \([N(P_5 \Phi_3, \Gamma_2) : N(P_5 \Phi_3, \Gamma_2) \cap \Gamma_0^2(4)] = 24\). This proves the case of \(\Phi_{3b}\).

Let \(N_3\) be the subgroup of \(N_1\) consisting of the elements such that \(a_1 + c_1 - d_1 \equiv c_1 + c_2 \equiv 0 \pmod{2}\). Then

\[
P_7 N(\Phi_3, \Gamma_2) P_7^{-1} \cap \Gamma_0^2(4) = P_7(\varepsilon N_3 \varepsilon^{-1}) P_7^{-1} \cup P_7(\delta \varepsilon \gamma N_3 \varepsilon^{-1}) P_7^{-1}
\]

and

\[
[P_7 N(\Phi_3, \Gamma_2) P_7^{-1} : P_7 N(\Phi_3, \Gamma_2) P_7^{-1} \cap \Gamma_0^2(4)] = [N_1 : N_3].
\]

Since \([N_3 : N_1(2)] = 3\), we have \([N(P_7 \Phi_3, \Gamma_2) : N(P_7 \Phi_3, \Gamma_2) \cap \Gamma_0^2(4)] = 2\). This proves the case of \(\Phi_{3c}\). \(\square\)
In the following theorem we list \( J(P\varphi P^{-1}, P\langle Z\rangle) \) and \( \psi(\det D) \), where \( \varphi \) is an element of \( C^p(\Phi, \Gamma_0^2(4N)) \) such that \( P\varphi P^{-1} \in C^p(\Phi, \Gamma_0^2(4N)) \), \( Z \in \Phi \) and \( D \) is the lower right \( 2 \times 2 \) matrix of \( P\varphi P^{-1} \). We list \( P\Phi, \varphi, J(P\varphi P^{-1}, P\langle Z\rangle) \) and \( \psi(\det D) \) in this order. In the case where \( \Phi \) is in the divisor at infinity, \( J(P\varphi P^{-1}, P\langle Z\rangle) \) means the limit of \( J(P\varphi P^{-1}, P\langle Z\rangle) \) when \( Z \) tends to \( \Phi \).

**Theorem 3.9.** The proper elements of the isotropy groups of the fixed points (sets) of \( \Gamma_0^2(4) \) and the values of the automorphy factor of weight \( 1/2 \) and the character \( \psi \) are as follows. We assume that \( r + t \equiv 0 \pmod{4} \) for the elements whose notations are marked by *1) and assume that \( r - t \equiv 0 \pmod{2} \) for the elements whose notations are marked by *2). The meaning of the mark * of \( \Phi \) is the same as in the above theorem.

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<td>( \varphi_{15}(r) )</td>
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<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Phi_{15b} )</td>
<td>( \varphi_{15}(4r) )</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Phi_{15c} )</td>
<td>( \varphi_{15}(r) )</td>
<td>( (i)^r )</td>
<td>( (-1)^r )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Phi_{16a}^* )</td>
<td>( \varphi_{16}(r) )</td>
<td>1</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\[
\begin{align*}
\Phi_{16b} & \quad \psi_{16}(4r) & \quad 1 & \quad -1 \\
\Phi_{16c} & \quad \psi_{16}(r) & \quad (i)^t & \quad (-1)^{r+1} \\
\Phi_{16d} & \quad \psi_{16}(r) & \quad (i)^t & \quad (-1)^{r+1} \\
\Phi_{16e} & \quad \psi_{16}(4r) & \quad 1 & \quad -1 \\
\Phi_{17a} & \quad \psi_{17}(r) & \quad 1 & \quad -1 \\
\Phi_{17b} & \quad \psi_{17}(r) & \quad (i)^t & \quad (-1)^{r+1} \\
\Phi_{17c} & \quad \psi_{17}(r) & \quad 1 & \quad -1 \\
\Phi_{17d} & \quad \psi_{17}(r) & \quad - (i)^t & \quad (-1)^{r+1} \\
\Phi_{17e} & \quad \psi_{17}(4r) & \quad 1 & \quad -1 \\
\Phi_{22a} & \quad \psi_{22}(1, r, t) & \quad 1 & \quad 1 \\
\Phi_{22b} & \quad \psi_{22}(3, r, t) & \quad 1 & \quad -1 \\
\Phi_{22c} & \quad \psi_{22}(1, 4r, 4t) & \quad 1 & \quad 1 \\
\Phi_{22d} & \quad \psi_{22}(3, 4r, 4t) & \quad 1 & \quad -1 \\
\Phi_{22e} & \quad \psi_{22}(1, r, t)^{+1} & \quad 1 & \quad 1 \\
\Phi_{22f} & \quad \psi_{22}(3, r, t)^{+1} & \quad 1 & \quad -1 \\
\Phi_{22g} & \quad \psi_{22}(1, 2r, 2t)^{+2} & \quad (-1)^t & \quad 1 \\
\Phi_{22h} & \quad \psi_{22}(3, 2r, 2t)^{+2} & \quad (-1)^t & \quad -1 \\
\Phi_{23a} & \quad \psi_{23}(2, r, t) & \quad 1 & \quad 1 \\
\Phi_{23b} & \quad \psi_{23}(4, r, t) & \quad 1 & \quad -1 \\
\Phi_{23c} & \quad \psi_{23}(2, 4r, 4t) & \quad 1 & \quad 1 \\
\Phi_{23d} & \quad \psi_{23}(4, 4r, 4t) & \quad 1 & \quad -1 \\
\Phi_{23e} & \quad \psi_{23}(4, r, t) & \quad (i)^t & \quad (-1)^{r+1} \\
\Phi_{23f} & \quad \psi_{23}(4, 4r, 4t) & \quad 1 & \quad -1 \\
\Phi_{23g} & \quad \psi_{23}(4, 4r, 4t) & \quad (i)^t & \quad (-1)^{r+1}
\end{align*}
\]
\[ \Phi_{23f}^* \quad \psi_{23}(2, r, t) \quad (i)^{r+t} \quad (-1)^{r+t} \]
\[ \psi_{23}(4, r, t) \quad (i)^{r+t} \quad (-1)^{r+t+1} \]
\[ \Phi_{23g} \quad \psi_{23}(2, 2r + 1, 2t + 1)^{(2)} \quad i(-1)^t - 1 \]
\[ \psi_{23}(4, 2r + 1, 2t + 1)^{(2)} \quad i(-1)^t + 1 \]
\[ \Phi_{24a}^* \quad \psi_{24}(2, r, t) \quad 1 \quad 1 \]
\[ \psi_{24}(4, r, t) \quad 1 \quad -1 \]
\[ \Phi_{24b}^* \quad \psi_{24}(4, r, t) \quad (i)^t \quad (-1)^{t+1} \]
\[ \Phi_{24c}^* \quad \psi_{24}(4, 4r, t) \quad 1 \quad -1 \]
\[ \Phi_{24d}^* \quad \psi_{24}(2, r, t) \quad - (i)^{r+t} \quad (-1)^{r+t} \]
\[ \psi_{24}(4, r, t) \quad - (i)^{r+t} \quad (-1)^{r+t+1} \]
\[ \Phi_{24e} \quad \psi_{24}(2, 2r, 2r)^{(2)} \quad (-1)^t \quad 1 \]
\[ \psi_{24}(4, 2r, 2r)^{(2)} \quad (-1)^t \quad -1 \]

(24)

(25)

\[ \Phi_{25a}^* \quad \psi_{25}(1, r, s, t) \quad 1 \quad 1 \]
\[ \psi_{25}(2, r, s, t) \quad 1 \quad 1 \]
\[ \psi_{25}(3, r, s, t) \quad 1 \quad 1 \]
\[ \psi_{25}(4, r, s, t) \quad 1 \quad -1 \]
\[ \psi_{25}(5, r, s, t) \quad 1 \quad -1 \]
\[ \psi_{25}(6, r, s, t) \quad 1 \quad -1 \]
\[ \Phi_{25b} \quad \psi_{25}(1, 4r, 4s, 4t) \quad 1 \quad 1 \]
\[ \psi_{25}(2, 4r, 4s, 4t) \quad 1 \quad 1 \]
\[ \psi_{25}(3, 4r, 4s, 4t) \quad 1 \quad 1 \]
\[ \psi_{25}(4, 4r, 4s, 4t) \quad 1 \quad -1 \]
\[ \psi_{25}(5, 4r, 4s, 4t) \quad 1 \quad -1 \]
\[ \psi_{25}(6, 4r, 4s, 4t) \quad 1 \quad -1 \]
\[ \Phi_{25c} \quad \psi_{25}(1, r, s, t) \quad (i)^t \quad (-1)^t \]
\[ \psi_{25}(6, r, s, t) \quad (i)^t \quad (-1)^{t+1} \]
\[ \Phi_{25d}^* \quad \psi_{25}(1, 4r, s, t) \quad 1 \quad 1 \]
\[ \psi_{25}(5, 4r, s, t) \quad 1 \quad -1 \]
\[ \Phi_{25e} \quad \psi_{25}(1, 4r, 2s, t) \quad (i)^t \quad (-1)^t \]
\[ \psi_{25}(5, 4r, 2s, t) \quad (i)^t \quad (-1)^{t+1} \]
Proof. Due to the transformation formula of $\Theta(Z)$ (Theorem 1.4). When $\Phi$ is in the divisor at infinity, $\phi$ includes parameters (for example “$r$” of $\phi_{15}(r)$). In such cases we have a problem to evaluate the Gaussian sum $\lambda(P\phi P^{-1})$. But we skip this problem as follows. Since $\phi_{15}(r) = \phi_{15}(1)r$, we have
\[
\lim_{Z \to \Phi_{15}} J(P\phi_{15}(r)P^{-1}, P\langle Z \rangle) = \lim_{Z \to \Phi_{15}} J(P\phi_{15}(1)P^{-1}, P\langle Z \rangle) r.
\]
Hence it suffices to compute $J(P\phi P^{-1}, P\langle Z \rangle)$ for the generators of $C(P\Phi, \Gamma_2^0(4)/\Gamma_2(4N))$.

4. The dimension formula

In this section we present the explicit dimension formulas and also prove $M_{2j,k+1/2}(\Gamma_2^0(4), \psi) = S_{j,k+1/2}(\Gamma_2^0(4), \psi)$. We can prove the following vanishing theorem similarly as in [T5], Theorem 6.1 by using the vanishing theorem of Kawamata-Viehweg ([Ka], [V]).

**Theorem 4.1.** If $j = 0$ and $k \geq 3$ or if $j \geq 1$ and $k \geq 4$, then
\[
H^i(\tilde{X}_2(4N), \mathcal{O}(\text{Sym}^i(\tilde{V})) \otimes \tilde{H}_2^{\otimes(2k+1)} \otimes [D]^{\otimes(-1)}) \cong [0]
\]
for $i > 0$.

By using this theorem and the theorem of Riemann-Roch-Hirzebruch we have

**Theorem 4.2.** If $j = 0$ and $k \geq 3$ or if $j \geq 1$ and $k \geq 4$, then
\[
S_{j,k+1/2}(\Gamma_2(4N)) = 2^{3-1}(j+1)(2(2k-3)(2j+2k-1)(j+2k-2)N^{10}
- 30(j+2k-2)N^8 + 45N^7) \times \prod_{p \text{ odd prime dividing } N}(1-p^{-2})(1-p^{-4})
\]
where $p$ is over odd prime numbers which divide $N$.

**Proof.** It suffices to replace $k$ and $N$ in the formula of the dimension of $S_{j,k}(\Gamma_2(N))$ ([T3]) with $k+1/2$ and $4N$, respectively.

To evaluate the sums which appear in the computation of Theorem 4.4 and Theorem 4.5 we use the following

**Lemma 4.3.** Let $\zeta = e(1/4N)$. Then we have
\[ \sum_{r=1}^{4N-1} \frac{(i)^{kr}}{(1 - \zeta r)} = \begin{cases} \frac{-1 - 4N}{2}, & \text{if } k \equiv 0 \pmod{4}, \\ \frac{-1 + 2N}{2}, & \text{if } k \equiv 1 \pmod{4}, \\ \frac{-1}{2}, & \text{if } k \equiv 2 \pmod{4}, \\ \frac{-1 - 2N}{2}, & \text{if } k \equiv 3 \pmod{4}. \end{cases} \]

\[ \sum_{r=1}^{4N-1} \frac{(i)^{kr}}{(1 - \zeta r)^2} = \begin{cases} \frac{-16N^2 + 24N - 5}{12}, & \text{if } k \equiv 0 \pmod{4}, \\ \frac{2N^2 - 12N - 5}{12}, & \text{if } k \equiv 1 \pmod{4}, \\ \frac{8N^2 - 5}{12}, & \text{if } k \equiv 2 \pmod{4}, \\ \frac{2N^2 + 12N - 5}{12}, & \text{if } k \equiv 3 \pmod{4}. \end{cases} \]

\[ \sum_{r=1}^{4N-1} \frac{(i)^{kr}}{(1 - \zeta r)^3} = \begin{cases} \frac{-16N^2 + 16N - 3}{8}, & \text{if } k \equiv 0 \pmod{4}, \\ \frac{4N^3 + 2N^2 - 8N - 3}{8}, & \text{if } k \equiv 1 \pmod{4}, \\ \frac{8N^2 - 3}{8}, & \text{if } k \equiv 2 \pmod{4}, \\ \frac{-4N^3 + 2N^2 + 8N - 3}{8}, & \text{if } k \equiv 3 \pmod{4}. \end{cases} \]

The dimension of \( S_{2j,k+1/2}(\Gamma_0^2(4)) \) is calculated as

\[ \sum_{\Phi} \sum_{P} \sum_{M} J(P M P^{-1}, P(Z))^{2k+1} \frac{\tau_0(M, \Phi)}{|N(P \Phi, \Gamma_0^2(4)/\Gamma_2(4N))|}, \]

where \( \Phi \) is over the 15 fixed points (sets) in §2, \( P \) is over the representatives of \( \Gamma_0^2(4)/\Gamma_2/N(\Phi, \Gamma_2) \), \( M \) is over \( C^P(\Phi) \cap P^{-1} \Gamma_0^2(4) P \) and \( Z \in \Phi \). We have

**Theorem 4.4.** If \( j = 0 \) and \( k \geq 3 \) or if \( j \geq 1 \) and \( k \geq 4 \), the dimension of \( S_{2j,k+1/2}(\Gamma_0^2(4)) \) is given by the following Mathematica function:

```mathematica
SiegelHalf[j_, k_] := Block[{a, ljk},
    mod[x_, y_] := Mod[x, y] + 1;
    a = (2*j + 1) * (4*j + 2*k - 1) * (j + k - 1) * (2*k - 3) / 2^5/3^2;
    a = a + (2*j + 1) * If[Mod[k, 2] == 0, 19 - 22*k - 22*j, 25 - 22*k - 22*j] / 2^6/3;
    a = a + 3*(2*j + 1) * If[Mod[k, 2] == 0, -1, 1] / 2^6;
    ];
```
\(* \text{contribution of } \psi_1 *\)
\(* \text{contribution of } \psi_{15}(r) *\)
\(* \text{contribution of } \psi_{22}(1, r, t) *\)
\(* \text{contribution of } \psi_{25}(1, r, s, t) *\)
\[ a = a + (4j + 2k - 1) \cdot (2k - 3) / 2^6; \]
\[ a = a + \text{If}[\text{Mod}[k, 2] == 0, 17 - 12k + 12j, 49 - 20k - 20j] / 2^6; \]
\(* \text{contribution of } \psi_2 *\)
\(* \text{contribution of } \psi_{16}(r) *\)
\(* \text{contribution of } \psi_{23}(4, r, t) *\)
\[ a = a + 7j \cdot (4j + 2k - 1) \cdot (2k - 3) / 2^6 / 3; \]
\[ a = a + (35 - 48k - 48j) / 2^5 / 3; \]
\[ a = a - 13 / 2^4 / 3; \]
\[ a = a + \text{If}[\text{Mod}[k, 2] == 0, 17, 35] / 2^6; \]
\(* \text{contribution of } \psi_3 *\)
\(* \text{contribution of } \psi_{22}(3, r, t) *\)
\(* \text{contribution of } \psi_{24}(4, r, t) *\)
\(* \text{contribution of } \psi_{25}(i, r, s, t) (i = 4, 5, 6) *\)
\[ ljk = \{1, -1\}; \]
\[ a = a + (j + k - 1) \cdot ljk[[\text{Mod}[j, 2]]] / 2^3; \]
\[ a = a + \text{If}[\text{Mod}[k, 2] == 0, 3, 1] \cdot ljk[[\text{Mod}[j, 2]]] / 2^4; \]
\(* \text{contribution of } \psi_4 *\)
\(* \text{contribution of } \psi_{23}(2, r, t) *\)
\[ a = a + \text{If}[\text{Mod}[k, 2] == 0, 3, 1] \cdot ljk[[\text{Mod}[j, 2]]] / 2^4; \]
\(* \text{contribution of } \psi_5 *\)
\(* \text{contribution of } \psi_{24}(2, r, t) *\)
\[ ljk = \{1, 0, -1\}; \]
\[ a = a + 2 \cdot ljk[[\text{Mod}[j, 3]]] \cdot (j + k - 1) / 3^2; \]
\[ a = a - ljk[[\text{Mod}[j, 3]]] / 2; \]
\(* \text{contribution of } \psi_6 *\)
\(* \text{contribution of } \psi_{25}(2, r, s, t) \text{ and } \psi_{25}(3, r, s, t) *\)
\[ ljk = \{2j + 1\} \cdot \{1, 0, -1\}, \{0, -1, 1\}, \{-1, 1, 0\}; \]
\[ a = a + ljk[[\text{Mod}[j, 3]], \text{Mod}[k, 3]]] / 2 / 3^2; \]
\(* \text{contribution of } \psi_{10}(1) \text{ and } \psi_{10}(2) *\)
\[ ljk = \{1, -2, 1\}, \{-2, 1, 1\}, \{1, 1, -2\}; \]
\[ a = a + ljk[[\text{Mod}[j, 3]], \text{Mod}[k, 3]]] / 2 / 3^2; \]
\(* \text{contribution of } \psi_{10}(4) \text{ and } \psi_{10}(5) *\)
\[ ljk = \{1, -2, 1\}; \]
\[ a = a - ljk[[\text{Mod}[j, 3]]] / 2 / 3^2; \]
The dimension of $S_{2,j,k}^{1/2}(\Gamma_0^2(4), \psi)$ is calculated as

$$\sum_{\Phi} \sum_p \sum_M J(PP^{-1}, \Phi(\mathbb{Z}))^{2k+1} \psi(\det D) \frac{\tau_0(M, \Phi)}{|N(P\Phi, \Gamma_0^2(4)/\Gamma_2(4N))|},$$

where $D$ is the lower right $2 \times 2$ matrix of $PP^{-1}$. We have

**Theorem 4.5.** If $j = 0$ and $k \geq 3$ or if $j \geq 1$ and $k \geq 4$, then the dimension of $S_{2,j,k+1/2}(\Gamma_0^2(4), \psi)$ is given by the following Mathematica function:

```mathematica
SiegelHalfpsi[j_, k_] := Block[{a, ljk},
    mod[x_, y_] := Mod[x, y] + 1;
    a = (2*j + 1) * (4*j + 2*k - 1) * (j + k - 1) * (2*k - 3) / 2^5 / 3^2;
    a = a + (2*j + 1) * If[Mod[k, 2] == 0, 25 - 22*k - 22*j, 19 - 22*k - 22*j] / 2^6 / 3;
    a = a - 3 * (2*j + 1) * If[Mod[k, 2] == 0, -1, 1] / 2^6;
    a = a - (4*j + 2*k - 1) * (2*k - 3) / 2^6;
    a = a - If[Mod[k, 2] == 0, 49 - 20*k - 20*j, 17 - 12*k - 12*j] / 2^6;
    (* contribution of $\phi_{12}$ *)
    (* contribution of $\phi_{15}(r)$ *)
    (* contribution of $\phi_{22}(1, r, t)$ *)
    (* contribution of $\phi_{25}(1, r, s, t)$ *)
    a = a - (4*j + 2*k - 1) * (2*k - 3) / 2^6;
    a = a - If[Mod[k, 2] == 0, 49 - 20*k - 20*j, 17 - 12*k - 12*j] / 2^6;
    (* contribution of $\phi_2$ *)
    (* contribution of $\phi_{16}(r)$ *)
    (* contribution of $\phi_{23}(4, r, t)$ *)
    a = a - 7 * (4*j + 2*k - 1) * (2*k - 3) / 2^6 / 3;
    a = a - (35 - 48*k - 48*j) / 2^5 / 3;
    a = a + 13 / 2^4 / 3;
    a = a - If[Mod[k, 2] == 0, 15, 7] / 2^6;
    a = a - If[Mod[k, 2] == 0, 3, 2] / 2^2;
    (* contribution of $\phi_3$ *)
    (* contribution of $\phi_{17}(r)$ *)
    (* contribution of $\phi_{22}(3, r, t)$ *)
    (* contribution of $\phi_{24}(4, r, t)$ *)
    (* contribution of $\phi_{25}(i, r, s, t)$ (i = 4, 5, 6) *)
    ljk = {1, -1};
    a = a + (j + k - 1) * ljk[[mod[j, 2]]] / 2^3;
    a = a - If[Mod[k, 2] == 0, 5, 3] * ljk[[mod[j, 2]]] / 2^4;
    (* contribution of $\phi_4$ *)
]
(* contribution of $\varphi_{23}(2,r,t)$ *)
a = a - If[Mod[k, 2] == 0, 1, 3] ljk[[mod[j, 2]]]/2^4;

(* contribution of $\varphi_5$ *)

(* contribution of $\varphi_{24}(2,r,t)$ *)
ljk = {1, 0, -1};
a = a + 2*ljk[[mod[j, 3]]]*(j+k-1)/3^2;

(* contribution of $\varphi_6$ *)

(* contribution of $\varphi_{25}(2,r,s,t)$ and $\varphi_{25}(3,r,s,t)$ *)
ljk = (2*j + 1)*{{1, 0, -1}, {0, -1, 1}, {-1, 1, 0}};
a = a + ljk[[mod[j, 3], mod[k, 3]]]/2/3^2;

(* contribution of $\varphi_{10}(1)$ and $\varphi_{10}(2)$ *)
ljk = {{1, -2, 1}, {-2, 1, 1}, {1, 1, -2}};
a = a - ljk[[mod[j, 3], mod[k, 3]]]/2/3^2;

(* contribution of $\varphi_{10}(4)$ and $\varphi_{10}(5)$ *)
ljk = {1, -2, 1};
a = a + ljk[[mod[j, 3]]]/2/3^2;

(* contribution of $\varphi_{12}$ *)

Return[a];

Now we prove

Theorem 4.6.

$M_{2j,k+1/2}(\Gamma_0^2(4), \psi) = S_{2j,k+1/2}(\Gamma_0^2(4), \psi)$.

Proof. Let $Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$ and $f \in M_{2j,k+1/2}(\Gamma_0^2(4), \psi)$. We have to prove that

$\lim_{\text{Im} Z_2 \to \infty} f \mid [\xi]_{2j,k+1/2}(Z) = 0$ (*)

for any $\xi \in p^{-1}(\Gamma_2)$. Let $P_i$ ($i = 1, 2, 3, 4$) be the matrices which correspond to the representatives of one-dimensionalcusps as before and let us recall that

$\varphi_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

To prove the assertion, it suffices to prove (*) for $\xi = (P, \phi(Z))$ such that $P$ is $P_1$, $P_2$, $P_3$ or $P_4$. Let $Z$ be as above. From $\varphi_2(Z) = Z$, we have

$P(Z) = P\varphi_2(Z) = (P\varphi_2P^{-1})P(Z)$.
for any $P$. Let $i = 1, 2$ or $3$. Then $P_i \varphi_2 P_i^{-1} = \varphi_2$. Hence we have
\[
f(P_i (Z)) = f((P_i \varphi_2 P_i^{-1}) P_i (Z))
\]
\[
= J(\varphi_2, P_i (Z))^{2k+1} \psi(-1) f(P_i (Z))
\]
\[
= -f(P_i (Z)).
\]
Therefore $f(P_i (Z)) = 0$. Next let $i = 4$. Then we have
\[
P_4 \varphi_2 P_4^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & -4 & 1 & 0 \\
4 & 0 & 0 & -1
\end{pmatrix}
\]
and $J(P_4 \varphi_2 P_4^{-1}, P_4 (Z))$ is identically equal to $1$. Therefore similarly as above we have $f(P_4 (Z)) = 0$.

**Remark 4.7.** Note that $f(P_i (Z))$ ($i = 1, 2, 3, 4$) is identically zero for $Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$. So it may be natural to ask whether for any $P \in \Gamma_2$, $f(P (Z))$ is identically zero or not. But this is not true in general. Let us recall that $\Phi_2$ is \(\left\{ \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \right\} \) and let $P_8$ be as before. Then $P_1, P_4$ and $P_8$ are the representatives of $\Gamma_0^2(4) \backslash \Gamma_2 / N(\Phi_2, \Gamma_2)$.

\[
P_8 \varphi_2 P_8^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & -2 & 1 & 0 \\
2 & 0 & 0 & -1
\end{pmatrix}
\]
does not belong to $\Gamma_0^2(4)$ but belongs to $\Gamma_2^a$ and $J(P_8 \varphi_2 P_8^{-1}, P_8 (Z))$ is identically equal to $1$. Therefore if $f(Z)$ belongs to $M_{2, j+1/2}(\Gamma_2^a, \psi)$, it holds that $f(P (Z)) = 0$ for any $P \in \Gamma_2$ and $Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$. ($\psi$ is extendable to a character of $\Gamma_2^a$ (cf. Lemma 3.7).)

5. **The case $j = 0$**

In this section we prove $\bigoplus_{k=0}^{\infty} M_{k+1/2}(\Gamma_0^2(4))$ and $\bigoplus_{k=0}^{\infty} M_{k+1/2}(\Gamma_0^2(4), \psi)$ are modules of rank one over the graded ring of the modular forms of integral weights.

**Proposition 5.1.**
\[
\sum_{k=0}^{\infty} \dim S_{k+1/2}(\Gamma_0^2(4)) t^k = \sum_{k=0}^{\infty} \text{SiegelHalf } [0, k] t^k + t^2
\]
\[
= \frac{2t^5 + 2t^6 - t^7 - 2t^8 - t^9 + t^{10}}{(1 - t)(1 - t^2)^2(1 - t^3)}.
\]
Proof. If $f(Z) \in S_{k+1/2}(\Gamma_0^2(4))$, then $f(Z)\Theta(Z)^2 \in S_{k+3/2}(\Gamma_0^2(4))$. Since $\dim S_{1/2}(\Gamma_0^2(4))$ is equal to $\text{SiegelHalf}[0, 3] = 0$, we have $S_{5/2}(\Gamma_0^2(4)) \simeq S_{3/2}(\Gamma_0^2(4)) \simeq S_{1/2}(\Gamma_0^2(4)) \simeq \{0\}$. But since $\text{SiegelHalf}[0, 2] = -1, \text{SiegelHalf}[0, 1] = 0$ and $\text{SiegelHalf}[0, 0] = 0$, we have the equality of the first line. 

Now we have

**Theorem 5.2.**

\[
\sum_{k=0}^{\infty} \dim M_{k+1/2}(\Gamma_0^2(4)) r^k
\]

\[
= \sum_{k=0}^{\infty} \dim S_{k+1/2}(\Gamma_0^2(4)) r^k + 3 \sum_{k=0}^{\infty} \dim S_{k+1/2}(\Gamma_0^1(4)) r^k + 4 \sum_{k=0}^{\infty} r^k - (3 + 3t + t^2)
\]

\[
= \frac{2t^5 + 2t^6 - t^7 - 2t^8 - t^9 + t^{10}}{(1-t)(1-t^2)^2(1-t^3)} + \frac{3(t^4 + t^5)}{(1-t)^2} + \frac{4}{(1-t)} - (3 + 3t + t^2)
\]

\[
= \frac{1}{(1-t)(1-t^2)^2(1-t^3)}.
\]

**Proof.** Recall that

\[
\varphi_{15}(r) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

and put $Z = \begin{pmatrix} Z_1 \\ 0 \\ Z_2 \end{pmatrix}$. From Theorem 3.9 (15) $\Phi_{15c}$ we have

\[
\lim_{\text{Im }Z_2 \to \infty} J(P_3 \varphi_{15}(r) P_3^{-1}, P_3(Z)) = (i)^r,
\]

where $i = \sqrt{-1}$. Hence if $f \in M_{k+1/2}(\Gamma_0^2(4))$ and $r$ is an odd integer, then we have

\[
\lim_{\text{Im }Z_2 \to \infty} f(P_3(Z)) = \lim_{\text{Im }Z_2 \to \infty} f(P_3(\varphi_{15}(r)(Z)))
\]

\[
= \lim_{\text{Im }Z_2 \to \infty} f((P_3 \varphi_{15}(r) P_3^{-1}) P_3(Z))
\]

\[
= \lim_{\text{Im }Z_2 \to \infty} J(P_3 \varphi_{15}(r) P_3^{-1}, P_3(Z))^{2k+1} f(P_3(Z))
\]

\[
= (i)^r (2k+1) \lim_{\text{Im }Z_2 \to \infty} f(P_3(Z)).
\]

Therefore $\lim_{\text{Im }Z_2 \to \infty} f(P_3(Z))$ and $\lim_{\text{Im }Z_2 \to \infty} f(\xi k + 1/2(Z))$ are identically 0 where $\xi = (P_3, \varphi(Z))$. Namely, the $\Phi$-operators to the one-dimensional cusp $C_3$ and to the zero-dimensional cusps $Q_3$, $Q_6$ and $Q_7$ are 0-maps.

Next we prove the surjectivity of the $\Phi$-operators to other cusps. In general the Eisenstein series of Klingen type of degree $g$ attached to a cusp form of degree $r$ and weight $k$
converges if \( k > g + r + 1 \) ([KL]). We define Eisenstein series of half integral weight in the following. In case \( k \) is a half integer, their convergence is also proved similarly as in the case of integral weight.

Let \( N(B_0, \Gamma_2) \) and \( N(B_1, \Gamma_2) \) be as in §2 and let \( P_i \) \( (i = 1, 2, 4, 5) \) be as in §2. Let \( \xi_i = (P_i, \phi_i(Z)) \in G_2 \) \( (i = 1, 2, 4, 5) \). We assume that \( \xi_1 = (1_4, 1) \) and \( \xi_4 = (P_4, J(P_4, Z)) \) since \( P_4 \in \Gamma_2^5 \). First we prove the case of zero-dimensional cusps. Let 1 be the function on \( \mathbb{S}_2 \) which is identically 1. Let

\[
E_i(Z) = \sum_\gamma 1 \left[ \xi_i^{-1}(\gamma) \right]_{k+1/2}(Z),
\]

where \( \gamma \) is over \( (P_i N(B_0, \Gamma_2) P_i^{-1} \cap \Gamma_0^2(4)) \setminus \Gamma_0^2(4) \). Let \( M \in N(B_0, \Gamma_2) \) and assume that \( P_i M P_i^{-1} \in \Gamma_0^2(4) \). We prove \( \xi_i (M) \xi_i^{-1} = \iota(P_i M P_i^{-1}) \) \( (i = 1, 2, 4, 5) \).

\[
1 \left[ \xi_i^{-1}(P_i M P_i^{-1}) \right]_{k+1/2}(Z) = (1 \left[ \iota(M) \right]_{k+1/2}) \left[ \xi_i^{-1}(\gamma) \right]_{k+1/2}(Z) = 1 \left[ \xi_i^{-1}(\gamma) \right]_{k+1/2}(Z).
\]

Therefore \( 1 \left[ \xi_i^{-1}(\gamma) \right]_{k+1/2}(Z) \) is independent of the choice of \( \gamma \).

We prove our assertion. The case of \( i = 1 \) or \( i = 4 \) is trivial. Similarly as in the proof of Theorem 1.8, we have

\[
\iota(P_i M P_i^{-1})(\xi_i (M) \xi_i^{-1})^{-1} = \iota(P_i M P_i^{-1}) \xi_i (M^{-1}) \xi_i^{-1} = (1_4, t),
\]

where

\[
t = J(P_i M P_i^{-1}, P_i M^{-1} P_i^{-1}(Z)) \phi_i (M^{-1} P_i^{-1}(Z)) J(M^{-1}, P_i^{-1}(Z)) \phi_i (P_i^{-1}(Z))^{-1}
\]

is a constant. We prove that \( t = 1 \). Let \( Z = P_i M(Z') \). Since \( J(M^{-1}, P_i^{-1}(Z)) = 1 \), \( t \) is equal to

\[
J(P_i M P_i^{-1}, P_i (Z')) \phi_i (Z') \phi_i (M(Z'))^{-1}.
\]

Let

\[
M_1 = \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}, \ S \in M(2, \mathbb{Z}), \ S = 'S \quad \text{and} \quad M_2 = \begin{pmatrix} U & O \\ O & t U^{-1} \end{pmatrix}, \ U \in GL(2, \mathbb{Z}).
\]

Let \( S = \begin{pmatrix} r & s \\ s & t \end{pmatrix} \) and \( U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Elements of \( N(B_0, \Gamma_2) \) have the form of \( M_1 M_2 \).

Let \( i = 2 \). Then \( P_2 M_1 M_2 P_2^{-1} \) belongs to \( \Gamma_0^2(4) \) if and only if \( r, s \) and \( t \) are divisible by 4. Since \( \lim_{Z \to \infty} \phi_2(Z') \phi_2(M_1(Z'))^{-1} = 1 \) (cf. Proof of Theorem 1.8), the assertion for \( M_1 \) follows from Theorem 3.9 (25) \( \Phi_{25} \phi_{25}(1, 4r, 4s, 4t) \). Since \( P_2 M_2 P_2^{-1} \in N(B_0, \Gamma_2) \), we have \( J(P_2 M_2 P_2^{-1}, P_2 (Z)) = 1 \). On the other hand we have

\[
\frac{\phi_2(Z')}{\phi_2(M_2(Z'))} = \frac{\sqrt{\det(-Z')}}{\sqrt{\det(-U Z' U)}} = 1.
\]
(It suffices to check in the case $Z$ is diagonal and $U$ is over the generators of $GL(2, \mathbb{Z})$.) So the assertion for $M_2$ was proved.

Let $i = 5$. $P_5 M_1 M_2 P_5^{-1}$ belongs to $\Gamma_0^2(4)$ if and only if $r$ and $b$ are divisible by 4. Since $\lim_{z \to \infty} \phi_5(Z)^{-1} = 1$, the assertion for $M_1$ is due to Theorem 3.9

\[(\text{25}) \Phi_{25}^r \varphi_{25}(1, 4r, s, t).\]

Let $i = 5$. $\tilde{\Gamma}_1$, $0(4) = \{(abcd) \in GL(2, \mathbb{Z}) \mid b \equiv 0 \pmod{4}\}$. It suffices to prove the assertion for them. If $U = U_2$ or $U = U_3$, the assertion is trivial since $\phi_5(Z) = \phi_5(M_2(Z))$ and $P_5 M_2 P_5^{-1} \in N(B_0, \Gamma_2)$. Let $U = U_1$ and $Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$.

Then

$$\frac{\phi_5(Z)}{\phi_5(M_2(Z))} = \frac{\sqrt{Z_1}}{\sqrt{Z_1 + 16Z_2}}.$$ 

We assume $\arg \sqrt{Z_1}$ is in $(0, \pi/2)$. Since $Z$ and $UZ^t U$ are connected by the path

$$\begin{pmatrix} Z_1 + t^2Z_2 \\ tZ_2 \\ Z_2 \end{pmatrix} (0 \leq t \leq 4)$$

which is on $\mathbb{S}_2$, $\arg \sqrt{Z_1 + 16Z_2}$ is also in $(0, \pi/2)$. On the other hand

$$P_5 M_2 P_5^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{pmatrix}.$$ 

From the transformation formula of $\Theta(Z)$ we have

$$J(P_5 M_2 P_5^{-1}, P_5(Z)) = \frac{\sqrt{Z_1 + 16Z_2}}{Z_1}.$$ 

Its argument is in $(-\pi/2, \pi/2)$ (cf. Remark 1.2). Hence the assertion was proved.

If $k \geq 3$, the series of $E_i(Z)$ ($i = 1, 2, 4, 5$) converges and $E_i(Z) \in S_{k+1/2}(\Gamma_0^2(4))$. Similarly as in the case of integral weight we can prove that $\lim_{z \to \infty} E_i \mid [\xi_j]_{k+1/2}(Z) = 1$ and $\lim_{z \to \infty} E_i \mid [\xi_j]_{k+1/2}(Z) = 0$ ($i \neq j$). Hence $\Phi$-operators to the zero-dimensional cusps $Q_1$, $Q_2$, $Q_4$, $Q_5$ are surjective if $k \geq 3$.

Next we construct Eisenstein series of Klingen type and prove the case of one-dimensional cusps. $M \in N(B_1, \Gamma_2)$ has the following form.
where $M_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2$, $m, n, r \in \mathbb{Z}$ and $u = \pm 1$. We denote the matrices of the right hand side by $M_1, M_2, M_3, M_4$, respectively. Let $Z = \begin{pmatrix} Z_1 & Z_{12} \\ Z_{12} & Z_2 \end{pmatrix}$. It is easily seen that

$$J(M, Z) = J(M_0, Z_1).$$

Let $\xi_i$ ($i = 1, 2, 4$) be as before. Let $f \in S_{k+1/2}(\Gamma_0^2(4))$. Since $S_{k+1/2}(\Gamma_0^2(4)) \simeq \{0\}$ ($k \leq 3$), we can assume that $k \geq 4$. Let $M \langle Z_1 \rangle$ be the upper-left entry of $M \langle Z \rangle$. We have $M \langle Z_1 \rangle = M_0 \langle Z_1 \rangle$. We put $\tilde{f}(Z) = f(Z_1)$. Then

$$\tilde{f} \mid [i(P_{\xi_i}^{-1} \gamma)]_{k+1/2}(Z) = f(M \langle Z \rangle)J(M, Z)^{-2k-1} = f(M \langle Z_1 \rangle)J(M_0, Z_1)^{-2k-1} \equiv f(Z_1) = \tilde{f}(Z).$$

Let $i = 1$ or $4$ and define

$$E_{i, f}(Z) = \sum_{\gamma} \tilde{f} | [i(P_{\xi_i}^{-1} \gamma)]_{k+1/2}(Z)$$

where $\gamma$ is over $(P, N(B_1, \Gamma_2)P_{\xi_i}^{-1} \cap \Gamma_0^2(4)) \setminus \Gamma_0^2(4)$. $\tilde{f} | [i(P_{\xi_i}^{-1} \gamma)]_{k+1/2}(Z)$ is independent of the choice of $\gamma$ from the above observation.

We return to the general case of degree $g$. Let

$$\Gamma^{g, 0}(4) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \mid B \equiv O \pmod{4} \right\}.$$

Then $aG_g\alpha^{-1} \cap \Gamma_g$ contains $\Gamma^{g, 0}(4)$. Let $\Theta^0(Z) = \theta(Z/2)$. If $M$ belongs to $\Gamma^{g, 0}(4)$, then

$$J^0(M, Z) := \Theta^0(M \langle Z \rangle)/\Theta^0(Z)$$

is holomorphic on $\mathcal{S}_g$. By using $J^0(M, Z)$ we define the space $S_{k+1/2}(\Gamma^{g, 0}(4))$ similarly as before. Let $Q_g = \begin{pmatrix} 4 & 1_g \\ 0 & 1_g \end{pmatrix}$ and $\lambda_g = (Q_g, 1) \in \bar{G}_g$. Let $M \in \Gamma^{g, 0}(4)$ and $\iota^0(M) = (M, J^0(M, Z))$. By definition we have $Q_g^{-1}J^{g, 0}(4)Q_g = J^{\iota^0}(4)$ and

$$J(Q_g^{-1}MQ_g, Q_g^{-1} \langle Z \rangle) = J^0(M, Z).$$

Hence it follows $\lambda_g^{-1}\iota^0(M)\lambda_g = \iota(Q_g^{-1}MQ_g)$. If $f \in S_{k+1/2}(\Gamma_0^2(4))$, then
(f | [λ_g^{-1}]_{k+1/2}) | [λ_g^{-1}]_{k+1/2}(Z) = (f | [λ_g^{-1}]_{k+1/2}(M) | [λ_g^{-1}]_{k+1/2}(Z) = f | [λ_g^{-1}]_{k+1/2}(Z).

Therefore f \mapsto f | [λ_g^{-1}]_{k+1/2} is an isomorphism of S_{k+1/2}(Γ^0(4)) to S_{k+1/2}(Γ^{0,0}(4)).

Let f \in S_{k+1/2}(Γ^0(4)) and \varnothing^0 = f | [λ_1^{-1}]_{k+1/2} \in S_{k+1/2}(Γ^{0,0}(4)). We put \tilde{f}^0(Z) = \varnothing^0(Z_1) for Z \in \mathbb{S}_Z. We have P_2 Γ^{2,0}(4)P_2^{-1} = Γ_0^2(4). Let

E_{2,f}(Z) = \sum_{γ} \tilde{f}^0(\xi^{-1}_2(γ))_{k+1/2}(Z)

where γ is over (P_2 N(B_1, Γ_0^2(4)) \cap Γ_0^2(4)) \backslash Γ_0^2(4). M \in N(B_1, Γ_2) is decomposed to a product M_1M_2M_3M_4 as before. We assume M belongs to Γ^{2,0}(4). Namely, b, n and r are divisible by 4. We prove \xi^0(M)ξ^{-1} = ι(P_2 MP_2^{-1}). Then \tilde{f}^0(\xi^{-1}(γ))_{k+1/2}(Z) is independent of the choice of γ since \tilde{f}^0(\xi^0(M))_{k+1/2}(Z) = \tilde{f}^0(Z).

Let Z = PM(Z'). Then

ι(P_2 MP_2^{-1})ξ^0(M)ξ^{-1} = ι(P_2 MP_2^{-1})ξ^0(M)ξ^{-1} = (1, t),

where

t = J(P_2 MP_2^{-1}, P_2(Z')) φ_2(Z') J^0(M, Z')^{-1} φ_2(M(Z'))^{-1}

is a constant. We prove that t = 1. Let Z' = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}. Then the case of M_3 is trivial.

Since J^0(M_4, Z') = 1 and lim_{mZ_2→∞}φ_2(Z')φ_2(M_4(Z'))^{-1} = 1, the assertion for M_4 is due to Theorem 3.9 (15) Φ_{15b}. The case of M_2 is easily proved if m = 1 and n = 0.

Let m = 0 and n = 4. Then J^0(M_2, Z') = 1. When W moves on the segment from Z' = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} to M_2(Z') = \begin{pmatrix} Z_1 & 4 \\ 4 & Z_2 \end{pmatrix}, det W moves on the segment from Z_1Z_2 to Z_1Z_2 - 16. Hence the argument of

\frac{φ_2(Z')}{φ_2(M_2(Z'))} = \sqrt{\frac{Z_1Z_2}{Z_1Z_2 - 16}}

is in (−π/2, π/2). On the other hand

P_2M_2P_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 0 & 1 \\ 0 & -4 & 1 & 0 \end{pmatrix}.

From the transformation formula of θ(Z) we have

J(P_2M_2P_2^{-1}, P_2(Z')) = \sqrt{\frac{Z_1Z_2 - 16}{Z_1Z_2}}.
Its argument is in \((-\pi/2, \pi/2)\). Hence the assertion was proved. Now we prove the case of \(M_1\). Since \(I^{1,0}(4)/(\pm 12)\) is generated by \(\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\) and \(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\), it suffices to prove the assertion for them. Let \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \). Then \(J^0(M_1, Z') = 1\).

\[
\frac{\phi_2(Z')}{\phi_2(M_1(Z'))} = \frac{\sqrt{Z_1 Z_2}}{\sqrt{(Z_1 + 4)Z_2}} \quad \text{and} \quad P_2 M_1 P_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

From the transformation formula of \(\Theta(Z)\) we have

\[
J(P_2 M_1 P_2^{-1}, P_2(Z')) = \frac{Z_1 + 4}{Z_1}.
\]

Hence the assertion is similarly proved as before. Let \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \). Then since \(P_2 M_1 P_2^{-1} \in N(B_0, \Gamma_2)\), \(J(P_2 M_1 P_2^{-1}, P_2(Z')) = 1\).

\[
\frac{\phi_2(Z')}{\phi_2(M_1(Z'))} = \frac{\sqrt{Z_1 Z_2}}{\sqrt{Z_1 Z_2/(Z_1 + 1)}}
\]

and

\[
J^0(M, Z') = J^0\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} . Z_1\right) = J\left(\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} . \frac{Z_1}{4}\right).
\]

This is calculated by the transformation formula and equal to \(\sqrt{Z_1 + 1}\). Hence the assertion is similarly proved as before.

Since \(k \geq 4\), the series of \(E_{i,f}(Z)\) \((i = 1, 2, 4)\) converges and \(E_{i,f}(Z) \in S_{k+1/2}(\Gamma_0^2(4))\). Similarly as in the case of integral weight we can prove that \(\lim_{Z_2 \to \infty} E_{i,f} | [\xi_i]_{k+1/2}(Z) = f(Z_1) (i = 1, 4)\), \(\lim_{Z_2 \to \infty} E_{2,f} | [\xi_2]_{k+1/2}(Z) = f^0(Z_2)\) and \(\lim_{Z_2 \to \infty} E_{i,f} | [\xi_i]_{k+1/2}(Z) = 0 \text{ (i \neq j)}\). Hence \(\Phi\)-operators to the one-dimensional cusps \(C_1, C_2\) and \(C_4\) are surjective. Now the theorem was proved for \(k \geq 3\).

We show that \(\dim M_{1/2}(\Gamma_0^2(4)) = 1\), \(\dim M_{3/2}(\Gamma_0^2(4)) = 1\) and \(\dim M_{5/2}(\Gamma_0^2(4)) = 3\). Then the first equality of the theorem is proved. Since \(\Theta(Z) \in M_{1/2}(\Gamma_0^2(4))\), \(\dim M_{1/2}(\Gamma_0^2(4)) \geq 1\). We have the product map:

\[
M_{1/2}(\Gamma_0^2(4)) \times M_{21/2}(\Gamma_0^2(4), \psi) \to M_{11}(\Gamma_0^2(4)) .
\]

Since \(\dim M_{21/2}(\Gamma_0^2(4), \psi) = \dim M_{11}(\Gamma_0^2(4)) = 1\) (cf. Proposition 5.3, and Proposition 5.4), \(\dim M_{1/2}(\Gamma_0^2(4)) = 1\). Similarly we have \(\Theta(Z)^3 \in M_{3/2}(\Gamma_0^2(4))\) and the product map:

\[
M_{3/2}(\Gamma_0^2(4)) \times M_{21/2}(\Gamma_0^2(4), \psi) \to M_{12}(\Gamma_0^2(4), \psi) .
\]
Since \( \dim M_{12}(\Gamma_0^2(4), \psi) = 1 \), we have \( \dim M_{3/2}(\Gamma_0^2(4)) = 1 \). Similarly we have the product maps:

\[
\begin{align*}
M_{5/2}(\Gamma_0^2(4)) \times M_{21/2}(\Gamma_0^2(4), \psi) & \to M_{13}(\Gamma_0^2(4)), \\
M_{1/2}(\Gamma_0^2(4)) \times M_{23/2}(\Gamma_0^2(4)) & \to M_{5/2}(\Gamma_0^2(4)).
\end{align*}
\]

Since \( \dim M_{13}(\Gamma_0^2(4)) = 3 \), we have \( \dim M_{5/2}(\Gamma_0^2(4)) \leq 3 \) and since \( \dim M_{3/2}(\Gamma_0^2(4)) = 3 \), we have \( \dim M_{5/2}(\Gamma_0^2(4)) \geq 3 \). Thus we have completed the proof of Theorem 5.2. \( \square \)

**PROPOSITION 5.3.**

\[
\sum_{k=0}^{0} \dim M_{k+1/2}(\Gamma_0^2(4), \psi)t^k = \sum_{k=0}^{0} \text{SiegelHalfpsi}[0, k]t^k + (3 + t + t^2)
\]

\[
= \frac{1}{(1-t)(1-t^2)^2(1-t^3)}.
\]

**Proof.** From Theorem 4.6, we have \( \dim M_{k+1/2}(\Gamma_0^2(4), \psi) = \dim S_{k+1/2}(\Gamma_0^2(4), \psi) \).

Since we have \( \dim S_{7/2}(\Gamma_0^2(4), \psi) = \text{SiegelHalfpsi}[0, 3] = 0 \), it follows that \( S_{5/2}(\Gamma_0^2(4), \psi) \simeq S_{3/2}(\Gamma_0^2(4), \psi) \simeq S_{1/2}(\Gamma_0^2(4), \psi) \simeq [0] \). But since we have \( \text{SiegelHalfpsi}[0, 2] = -1 \), \( \text{SiegelHalfpsi}[0, 1] = -1 \) and \( \text{SiegelHalfpsi}[0, 0] = -3 \), we have the equality of the first line. \( \square \)

Let \( M(\Gamma_0^2(4)) \), \( M(\Gamma_0^2(4), \psi) \) and \( A(\Gamma_0^2(4), \psi) \) be \( \bigoplus_{k=0}^{\infty} M_{k+1/2}(\Gamma_0^2(4)) \), \( \bigoplus_{k=0}^{\infty} M_{k+1/2}(\Gamma_0^2(4), \psi) \) and \( \bigoplus_{k=0}^{\infty} A_k(\Gamma_0^2(4), \psi^k) \), respectively. Then \( A(\Gamma_0^2(4), \psi) \) is a graded ring and since it holds \( J(M, Z)^2 = \det(CZ + D)\psi(\det D) \), \( M(\Gamma_0^2(4)) \) and \( M(\Gamma_0^2(4), \psi) \) are \( A(\Gamma_0^2(4), \psi) \)-modules. From the result of J.-I. Igusa ([Ig1]), we have the following proposition. (We can also prove them by dimension formula.)

**PROPOSITION 5.4.**

\[
\begin{align*}
\sum_{k=0}^{\infty} \dim M_k(\Gamma_0^2(4))t^k = & \frac{1 + t + t^3 + t^5}{(1-t^2)^3(1-t^6)}, \\
\sum_{k=0}^{\infty} \dim M_k(\Gamma_0^2(4), \psi)t^k = & \frac{1 + t + t^3 + t^5}{(1-t^2)^3(1-t^6)}, \\
\sum_{k=0}^{\infty} \dim M_k(\Gamma_0^2(4), \psi^k)t^k = & \frac{1 + t + t^3 + t^5}{(1-t^2)^3(1-t^6)}.
\end{align*}
\]

From this, Theorem 5.2 and Proposition 5.3, we have

**COROLLARY 5.5.** \( M(\Gamma_0^2(4)) \) and \( M(\Gamma_0^2(4), \psi) \) are free \( A(\Gamma_0^2(4), \psi) \)-modules of rank one.
A generator of $M(\Gamma_0^2(4))$ as $A(\Gamma_0^2(4), \psi)$-module is given by $\Theta(Z)$. Let $f_{21/2}(Z)$ be a generator of $M(\Gamma_0^2(4), \psi)$. Then $f_{21/2}(Z)\Theta(Z)$ is an automorphic form with respect to $J(M, Z)^{22}\psi(\text{det} D) = \text{det}(CZ + D)$\(^1\). Hence this belongs to $M_{11}(\Gamma_0^2(4))$. Let $f_{11}(Z)$ be a generator of $M(\Gamma_0^2(4), \psi)$. Then $f_{21/2}(Z)\Theta(Z)$ is an automorphic form with respect to $J(M, Z)^{22}\psi(\text{det} D) = \text{det}(CZ + D)$\(^1\). Hence this belongs to $M_{11}(\Gamma_0^2(4))$. Let $Z \in \mathcal{S}_2$. Then there exists $M \in \Gamma_2$ such that $M\langle Z \rangle = (Z_1 0 0 Z_2)$, if and only if one of ten theta constants vanishes at $Z$ (J.-I. Igusa, [H]). Hence $f_{21/2}(Z)$ does not belong to $S_{21/2}(\Gamma_0^2, \psi)$ (cf. Remark 4.7).

**Remark 5.6.** T. Ibukiyama represented the generators of $A(\Gamma_0^2(4), \psi)$ and $f_{21/2}(Z)$ explicitly by theta constants ([Ib]). Especially $A(\Gamma_0^2(4), \psi)$ is generated by algebraically independent modular forms $f_1, X, g_2$ and $f_3$ whose weights are $1, 2, 2$ and $3$, respectively. $f_{21/2}(Z)$ is divisible by nine theta constants and not divisible by one theta constant. Let $Z \in \mathcal{S}_2$. Then there exists $M \in \Gamma_2$ such that $M\langle Z \rangle = (Z_1 0 0 Z_2)$, if and only if one of ten theta constants vanishes at $Z$ (J.-I. Igusa, [H]). Hence $f_{21/2}(Z)$ does not belong to $S_{21/2}(\Gamma_0^2, \psi)$ (cf. Remark 4.7).

**Appendix. The generating functions**

We list here the generating functions of $\text{SiegelHalf}[j,k]$ and $\text{SiegelHalfpsi}[j,k]$.

**Table A.1.** $\sum_{j, k=0}^{\infty} \text{SiegelHalf}[j,k]s^jt^k$ is a rational function of $s$ and $t$ whose denominator is

$$(1-s^2)^2(1-s^3)^2(1-t^2)^2(1-t^3).$$

The coefficients of $s^jt^k$ ($0 \leq j \leq 9, 0 \leq k \leq 7$) in the numerator are given by the following matrix.

$$
\begin{array}{ccccccccc}
0 & 0 & -3 & -6 & -6 & -3 & 4 & 3 & -3 & -4 \\
0 & 0 & 1 & 1 & 1 & 3 & 3 & 1 & 1 & 1 \\
-1 & -1 & 7 & 17 & 20 & 8 & -12 & -8 & 8 & 10 \\
1 & 1 & 2 & 7 & 7 & -2 & -9 & -4 & 1 & 2 \\
2 & 3 & -2 & -12 & -20 & -9 & 8 & -4 & -8 & -8 \\
1 & 3 & -5 & -21 & -23 & -5 & 12 & 6 & -7 & -9 \\
0 & 0 & -1 & -1 & 2 & 2 & 1 & 3 & 4 & 2 \\
-2 & -3 & 4 & 14 & 13 & 0 & -8 & -2 & 7 & 7
\end{array}
$$

**Table A.2.** $\sum_{j, k=0}^{\infty} \text{SiegelHalfpsi}[j,k]s^jt^k$ is a rational function of $s$ and $t$ whose denominator is

$$(1-s^2)^2(1-s^3)^2(1-t^2)^2(1-t^3).$$
The coefficients of \( s_j^k \) \((0 \leq j \leq 9, \ 0 \leq k \leq 7)\) in the numerator are given by the following matrix:

\[
\begin{array}{cccccccc}
-3 & 0 & 6 & 6 & -6 & -21 & -11 & 3 & 6 & 2 \\
2 & 0 & -4 & -5 & 1 & 12 & 10 & 1 & -3 & -2 \\
6 & 0 & -12 & -11 & 17 & 47 & -6 & -12 & -4 \\
0 & 0 & 0 & 5 & 10 & 4 & -5 & -6 & -3 & 1 \\
-5 & 0 & 13 & 15 & -12 & -41 & -25 & -1 & 9 & 5 \\
-6 & 1 & 15 & 9 & -21 & -46 & -24 & 6 & 14 & 4 \\
3 & 2 & -6 & -12 & -3 & 13 & 14 & 6 & -2 & -3 \\
4 & 0 & -9 & -8 & 8 & 26 & 17 & 0 & -6 & -2
\end{array}
\]

References


[Ib] T. Ibukiyama, On Siegel modular forms of half integral weight of \( \Gamma_0(4) \) of degree two, (in preparation).


Siegel Cusp Forms of Half Integral Weight and Degree Two


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