PDE (Math 4163) Spring 2016

Section 3. PDE's in Unbounded Domains.

Firstly, we derive the vibrating string equation from Hamilton's principle: Suppose that $K(U)$ and $P(U)$, respectively, are the kinetic energy and potential energy of moving physical system described by $u = u(t)$.

Hamilton's principle states that a real movement u in the time interval $[t_1, t_2]$ is a zero of the first variation of the functional

$$J(v) = \int_{t_i}^{t_l} K(v(t)) - P(v(t))\, dt$$

where v is any possible movement of the system in the time interval $[t_1, t_2]$ with the same position at t_1 and t_2 as u i.e.:

$$v(t_1) = u(t_1)\,, \quad \text{and} \quad v(t_2) = u(t_2)\,.$$

Remarks:

1) In general u might be vector valued function.
2) Recall u is a zero of the first variation if

$$\lim_{\epsilon \to 0} \frac{d}{d\epsilon} J(u + \epsilon\phi) = 0\,.$$

for all $\phi = u - v$, and v a possible movement.

For an vibrating string over an interval $L = [a, b]$ with mass density ρ and tension coefficient μ the functional, J is given by

$$J(u) = \frac{1}{2} \int_{t_1}^{t_2} \int_L \rho\, u_t^2 d\, x\, dt - \frac{1}{2} \int_{t_1}^{t_2} \int_L \mu u_x^2\, dx\, dt.$$

Now

$$\frac{d}{d\epsilon}\Big(\frac{1}{2} \int_{t_1}^{t_2} \int_L \rho((u + \epsilon\phi)_t)^2\, dx\, dt\; - \frac{1}{2} \int_{t_1}^{t_2} \int_L \mu((u + \epsilon\phi)_x)^2\, dx\, dt\Big)$$

$$= \int_{t_1}^{t_2} \int_L \rho(u + \epsilon\phi)_t \cdot \phi_t \, dx \, dt \; - \int_{t_1}^{t_2} \int_L \mu(u_x + \epsilon\phi_x)\phi_x \, dx \, dt$$

For $\epsilon \to 0$, we obtain

$$0 = \int_{t_1}^{t_2} \int_L \rho(u_t)\phi_t \, dx \, dt - \int_{t_1}^{t_2} \int_L \mu \, u_x\phi_x \, dx \, dt.$$

Integrating by parts we get (for constant ρ and μ and $L = [a, b]$):

$$\int_{t_1}^{t_2} \int_L \rho u_{tt}\phi \, dx \, dt - \int_{t_1}^{t_2} \int_L \mu \, u_{xx}\phi \, dx \, dt$$

$$= \int_{t_1}^{t_2} \mu \, u_x\phi \Big|_a^b \, dt$$

since by assumption we have $\phi(x, t_1) = \phi(x, t_2) = 0$.

If the string is fixed on both ends we have

$$u(a, t) = v(a, t) \;\; u(b, t) = v(b, t),$$

for all possible movements v, i.e.:

$$\phi(a, t) = \phi(b, t) = 0, \text{ for all } t \in (t_1 t_2).$$

Hence, with $c^2 = \frac{\mu}{\rho}$ we get

$$(*) \qquad \int_{t_1}^{t_2} \int_L (u_{tt} - c^2 u_{xx})\phi \, dx \, dt = 0,$$

for all possible ϕ.

As a consequence of the Main Theorem of the Calculus of Variations we must have

$$u_{tt} - c^2 u_{xx} = 0.$$

which justifies the name vibrating string equation for this hyperbolic second order PDE.

Remark:

It is interesting to see what we get if we use u_t as a test function for this equation. Firstly, we have

$$\int_L u_{tt} \cdot u_t - c^2 u_{xx}\, u_t \; dx\; dt = 0\,.$$

Integrating the second term by parts yields

$$\int_L u_{tt} u_t^2) + c^2 u_x u_{xt}\; dx = c^2 (u_x \cdot u_t(x,t))\Big|_{x=a}^{x=b}\,.$$

If again $u(a,t)$ and $u(b,t)$ are fixed for all t we have $u_t(a,t) = u_t(b,t) = 0$ for all t, and obtain

$$\frac{d}{dt}(\frac{1}{2}\int_L u_t^2 + c^2\; u_x\; dx) = 0\,,.$$

After multiplying with 2ρ that yields

$$\frac{d}{dt}(K(t) + P(t)) = 0\,.$$

The last equation constitutes the Law of Conservation of Energy.

Solutions for the wave equation

1) Firstly, we recall from Chapter 2 d'Alembert's solution for the Cauchy problem of the wave equation,

$$(3.1)\qquad \begin{cases} u_{tt} - c^2\; u_{xx} = 0\,, & (x,t) \in (-\infty,\infty)\times(-\infty,\infty)\,, \\ u(x,0) = f(x)\,, \\ u_t(x,0) = g(x)\,. \end{cases}$$

It is given by

$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(r)\, dr\,.$$

2) Next we consider the inhomogeneous equation.

$$(3.2) \quad \begin{cases} u_{tt} - c^2\ u_{xx} = f(x,t) \\ u(x,0) = 0 \\ u_t(x,0) = 0\,. \end{cases}$$

In order to solve this we recall some facts from ODE. A particular solution of the inhomogeneous first order ODE

$$\begin{cases} w'(t) - c^2\ w(t) = f(t) \\ w(0) = 0\,, \end{cases}$$

is given by

$$w(t) = \int_0^t e^{c^2(t-\tau)} f(\tau) d\tau\,.$$

We note that

$$v(t,\tau) = e^{c^2(t-\tau)} f(\tau)$$

is a solution of the homogeneous ODE with an auxilliary condtion

$$\begin{cases} u' - c^2\ u = 0\,, \\ u(\tau) = f(\tau). \end{cases}$$

Trying to generalize this approach we might consider

$$\begin{cases} u_{tt} - c^2\ u_{xx} = 0\,, \\ u(x,\tau) = 0\,, \\ u_t(x,\tau) = f(x,\tau)\,, \end{cases}$$

which has the solution

$$v(x,t,\tau) = \frac{1}{2c} \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(r,\tau)\ dr\,.$$

Accordingly, we make the following guess for the solution of the inhomogeneous wave equation:

$$u(t,x) = \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(r,\tau)\ dr\ d\tau.$$

To verify, that our guess is indeed a solution of the inhomogeneous problem, we calculate

$$u_t = \frac{1}{2c} \int\limits_{x-0}^{x+0} f(r,t)\ dr$$

$$+\frac{1}{2c}\ c \int\limits_0^t f(x+c(t-\tau),\tau) + f(x-c(t-\tau),\tau)\ dt$$

and

$$u_{tt} = \frac{c}{2c}(f(x,t) + f(x,t))$$

$$+\frac{c^2}{2c} \int\limits_0^t f_r(x+c(t-\tau),\tau) - f_r(x-c(t-\tau),\tau)\ d\tau$$

$$u_x = \frac{1}{2c} \int\limits_0^t f(x+c(t-\tau),\tau) - f(x-c(t-\tau),\tau)\ d\tau$$

$$u_{xx} = \frac{1}{2c} \int\limits_0^t f_r(x+c(t-\tau),\tau) - f_r(x-c(t-\tau),\tau)\ d\tau$$

Hence

$$u_{tt} - c^2\ u_{xx} = f(x,t),$$

showing $u(x,t)$ is indeed a solution of 2).

(Here we used the notation $f_r(r,\tau) = (\frac{\partial}{\partial r} f)(r,\tau)$.)

Solving the inhomogeneous equation using an integral over the solution of a homogeneous equation is sometimes called Duhamel's Principle.

3) Superposing the solutions of 1) and 2), i.e.: defining

$$u = u_1 + u_2$$

for solutions of (3.1) and (3.2), respectively, we are able to deal with a problem in which all equations have an inhomogeneous term

$$\begin{cases} u_{tt} - c^2\, u_{xx} = f(x,t), \\ u(x,0) = \phi(x), \\ u_t(x,0) = \psi(x). \end{cases}$$

We get the solution

$$u(t,x) = \frac{1}{2c} \int_{\phi}^{t} \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(r,\tau)\, dr\, d\tau + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(r)\, dr$$

$$+\frac{1}{2}\phi(x+ct) + \phi(x-ct)\,.$$

Problems on the half line:

Firstly, we introduce the method of reflection. considering the initial boundary value problem with homogeneous B.C.

$$(IBVP) \begin{cases} u_{tt} - c^2\, u_{xx} = 0,\,, \quad (x,t) \in (0,\infty) \times (-\infty,\infty) \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \\ u(0,t) = 0 \end{cases}$$

In order to solve the IBVP we consider d'Alembert's solution

$$u(x,t) = \frac{1}{2}\tilde{\phi}(x-ct) + \frac{1}{2}\tilde{\phi}(x+ct)$$

of the initial value problem

$$u(x,0) = \tilde{\psi}, \text{ and } u_t(x,0) = 0.$$

If the incoming wave and the outgoing wave cancel at $x = 0$, then the B.C. would be satisfied. That is the case if

$$\tilde{\phi}(x+ct)|_{x=0} = \; -\tilde{\phi}(x-ct)|_{x=0} \text{ for all } t$$

or

$$\tilde{\phi}(r) = \; -\tilde{\phi}(-r) \text{ for all } r \in \mathbb{R}.$$

So, if $\tilde{\phi}(r)$ is the odd extension of ϕ, i.e.

$$\tilde{\phi}(r) = \begin{cases} \phi(r) & \text{if} \quad r \geq 0\,, \\ -\phi(-r) & \text{if} \quad r < 0\,. \end{cases}$$

then

$$u(x,t) = \frac{1}{2}\tilde{\phi}(x+ct) + \frac{1}{2}\tilde{\phi}(x-ct)$$

is a solution of

$$\begin{cases} u_{tt} - c^2\, u_{xx} = 0, & x \in (0,\infty)\ \times (-\infty,\infty) \\ u(x,0) = 0\,, \\ u(0,t) = \phi(x)\,, \\ u_t(0,t) = 0\,, \end{cases}$$

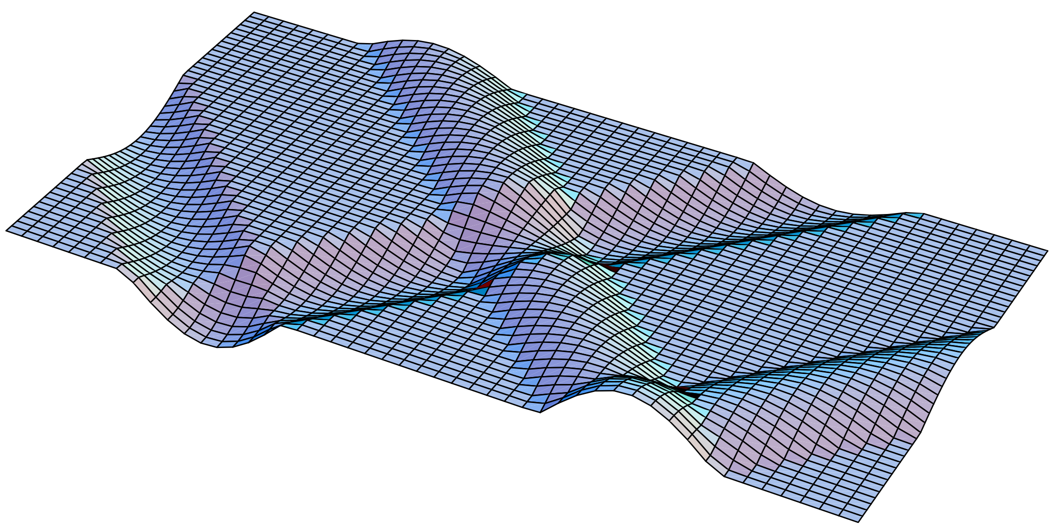

Fig. 5 : D'Alembert Solution for the odd extension

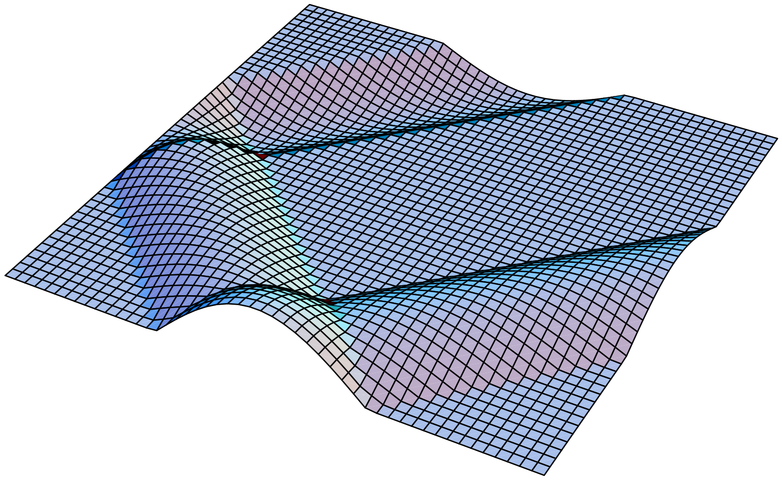

Fig. 6 : Solution of the IBVP

If we look at the d'Alembert's solution for the initial conditions

$$u(x,0) = 0\,, \qquad u_t(x,0) = \phi(x)\,,$$

that is

$$u(x,t) = \frac{1}{2c}\int\limits_{x-ct}^{x+ct} \phi(r)\; dr\,,$$

then for $x = 0$ we get

$$u(0,t) = \frac{1}{2c}\int\limits_{-ct}^{+ct} \phi(r)\; dr\,,$$

and we have $u(0,t) = 0$, if $\phi(r)$ is an odd function. Hence with $\tilde{\phi}(r)$ the odd extension of function $\phi(r)$ defined for $r \geq 0$ we get that

$$u(x,t) = \frac{1}{2c}\int\limits_{x-ct}^{x+ct} \tilde{\phi}(r)\; d\,r$$

is a solution of

$$\begin{cases} u_{tt} - c^2\, u_{xx} = 0, \\ u(x,0) = 0, \\ u_t(x,0) = \phi(x), \\ u(0,t) = 0\,. \end{cases}$$

Superposing those two solutions we note that

$$u(x,t) = \frac{1}{2}\phi(x+ct) + \tilde{\phi}(x-ct) + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} \tilde{\psi}(r)\, dr$$

is a solution of

$$\begin{cases} u_{tt} - c^2\, u_{xx} = 0 \\ u(0,t) = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x). \end{cases}$$

Likewise for the solution of an inhomogeneous PDE with homogeneous initial condition we have

$$u(x,t) = \frac{1}{2c}\int\limits_{0}^{t}\int\limits_{x-c(t-\tau)}^{x+c(t-\tau)} f(r,\tau)\, dr\, d\tau$$

in order for $0 = u(0,t)$ we need

$$0 = \frac{1}{2c}\int\limits_{0}^{t}\int\limits_{x-c(t-\tau)}^{x+c(t-\tau)} f(r,\tau)\, dr\, d\tau$$

and if $f(r,\tau)$ is an odd function in r then we have the desired equality. Consequently a solution of

$$\begin{cases} u_{tt} - c^2\, u_{xx} = f(x,t) \;\; (x,t) \in (0,\infty) \text{ x } (0,\infty) \\ u(0,t) = 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

is given by

$$u(x,t) = \frac{1}{2c}\int\limits_{0}^{t}\int\limits_{x-c(t-\tau)}^{x+c(t-\tau)} \tilde{f}(r,\tau)\, dr\, d\tau$$

with

$$\begin{cases} \tilde{f}(x,t) = f(x,t)\,, & \text{if} \quad x \geq 0\,, \\ -f(-x,t)\,, & \text{if} \quad x < 0\,, \end{cases}$$

"Reflection principle"

Remark
The method of defining the odd (or even) extensions of the data and then look for solutions on the whole line is sometimes called reflection principle.

Inhomogeneous B.C.

Considering

$$\begin{cases} u_{tt} - c^2\, u_{xx} = 0 \\ u(0,t) = g(t) \\ u(x,0) = u_t(x,0) = 0. \end{cases}$$

we introduce

$$v(x,t) = u(x,t) - g(t)$$

then v satisfies

$$\begin{cases} v_{tt} - c^2\, v_{xx} = -g''(t)\,, \\ v(0,t) = 0\,, \\ v(x,0) = -g(0)\,, \\ v_t(x,0) = -g'(0)\,, \end{cases}$$

this is a problem we just learned to handle.

Reflection principle on bounded intervals

We conclude the discussion on hyperbolic problems with some remarks to the application of the reflection principle to solve initial boundary value problems for the wave equation.

We just consider a vibrating string fixed at the end points at rest at time zero.

$$\begin{cases} u_{tt} - c^2\, u_{xx} = 0\,, \quad (x,t) \in (0,a) \times (-\infty, \infty)\,, \\ u(0,t) = 0\,, \\ u(a,t) = 0\,, \\ u(x,0) = \phi(x)\,, \\ u_t(x,0) = 0\,. \end{cases}$$

First we recall that a function $f(r)$ called point-symmetric at a point a (or anti-symmetric and sometimes just odd at a) if

$$f(a+x) = -f(a-x)\,,$$

that can be rewritten as

$$f(x) = -f(2a-x).$$

Note that the sin is point symmetric at each multiple of π. If we have a initial function $\hat{\phi}$, given for the wave equation on the entire line which is point symmetric at a, then the solution is zero at a for all time. We have

$$u(a,t) = \frac{1}{2}(\hat{\phi}(a-ct) + \hat{\phi}(a+ct)) = 0\,. \qquad (*)$$

That is $u(x,t)$ solves the boundary condition at a.

So in order to solve the boundary value problem we first continue ϕ to a function point symmetric at a:

$$\tilde{\phi} = \begin{cases} \phi(x)\,, & \text{for } x \in (0,a)\,; \\ -\phi(2a-x)\,. & \text{for } x \in (a,2a)\,; \end{cases}$$

Now we claim that if we continue $\tilde{\phi}$ periodically (with period $2a$) to a function $\hat{\phi}$ defined on the entire line, then

$$u(x,t) = \frac{1}{2}(\hat{\phi}(x-ct) + \hat{\phi}(x+ct)) = 0\,. \qquad (*)$$

is a solution of the (IBVP).

To define the extension $\hat{\phi}$ we note that for a given point x on the line there is integer k_x such that $(x - k_x 2a) \in [0, 2a)$ and we define

$$\hat{\phi}(x) = \tilde{\phi}(x - k_x 2a)\,.$$

To prove the claim we need to verify that $\hat{\phi}$ is an odd function and point symmetric in a.

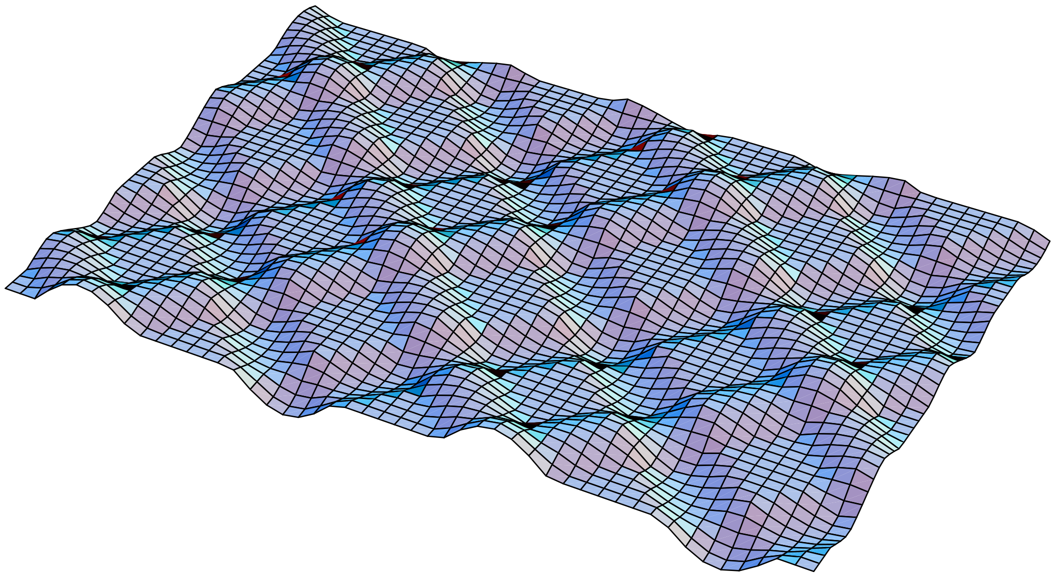

Fig. 7 : D'Alembert Solution for the extension

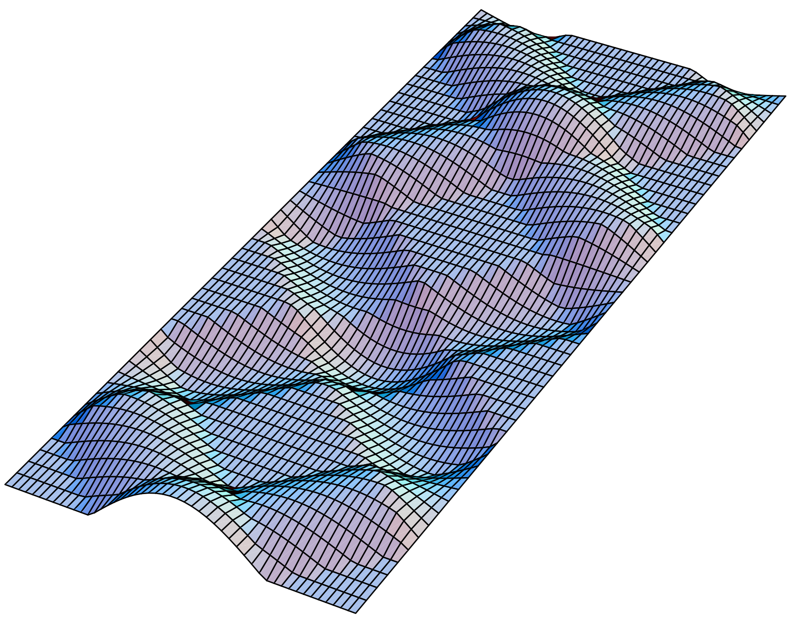

Fig. 8 : Solution of the IBVP

Use of Laplace transform to deal with hyperbolic PDE on the half line:

Firstly, we recall some facts of Laplace transformsx

Definition

Let f be a function defined on the positive real axis then we define for $s > 0$ the function

$$g(s) = \int_O^\infty f(t)e^{-st}dt$$

The function g is called the Laplace Transform and we write

$$g(s) = \mathcal{L}\{f(t)\}(s)$$

Examples:

1) $\mathcal{L}\{1\}(s) \ = \frac{1}{s}$,

2) $\mathcal{L}\{H_a(t)\}(s) \ = \frac{e^{-as}}{s}$,

where $H_a(t) = H(t-a)$ and H is the Heaviside function given by

$$H(t) = \begin{cases} 0, & \text{for } t < 0\,, \\ 1, & \text{for } t \geq 0 \end{cases}$$

Definition

A function f is called of exponential order for large t or as $t \to \infty$, if there are numbers c, M and T, such that

$$|f(t)| \leq Me^{ct}, \text{ for all } t \geq T\,.$$

We have the following

Theorem

If f is of exponential order and intergrable over all finite positve intervals (for intance piecewise continuous for $x \geq 0$,) then the Laplace transfrom exist for $s > c$.

Important for the application is the following

Corollary

If f is of exponential order then

$$\lim_{s \to \infty} \mathcal{L}\{f\}(s) \ \to 0\,.$$

Note $\int_0^\infty e^{-st}e^{ct}dt = \int_0^\infty e^{-(s-c)t}dt = \frac{1}{s-c}e^{-(s-c)t}$.

So if the absolute value of the function f is bounded by Me^{ct} we must have that the limit for $s \to \infty$ is zero.

The above calculation also gives

$$\mathcal{L}\{e^{at}\}(s) \;=\frac{1}{s-a}\,, \quad \text{for} \quad s > a$$

Laplace transform of powers $\alpha > -1$ of t :

$$\mathcal{L}\{t^{a}\}(s) \;=\int_0^\infty e^{-st}t^{\alpha}\;dt$$

using the substiution $\tau = st$ we have $\frac{d}{dt}\tau = ts\,.$ So

$$\mathcal{L}\{t^{a}\}(s) \;=\frac{1}{s}\frac{1}{s^{\alpha}}\int_0^\infty e^{-st}(ts)^{\alpha}s\,dt = \frac{1}{s^{\alpha+1}}\int_0^\infty e^{-\tau}(\tau)^{\alpha}\,,d\tau$$

We get

$$\mathcal{L}\{t^{a}\}(s) \;=\frac{\Gamma(a+1)}{s^{a+1}}\,,$$

where the Gamma function Γ is given by

$$\Gamma(r) = \int_0^\infty e^{-t}t^{(r-1)}dt$$

has the properties

$$\Gamma(r+1) = r\Gamma(r)\,.$$

Since $\Gamma(1) = 1\,,$ we get for nonngeative integers

$$\Gamma(n+1) = n!$$

$$\mathcal{L}\{t^{n}\}(s) \;=\frac{n!}{s^{n+1}}\,,$$

Theorem
The Laplace transform operates linearly on functions.

Example: Laplace transform of hypberbolic functions

$$\mathcal{L}\{\cosh kt\}(s) \;=\frac{1}{2}(\mathcal{L}\{e^{kt}\}(s) + \mathcal{L}\{e^{-kt}\}(s)\,) = \frac{1}{2}(\frac{1}{s-k}+\frac{1}{s+k})$$

$$= \frac{s}{s^2-k^2}\,.$$

Likewise we get

$$\mathcal{L}\{\sinh kt\}(s) \ = \frac{k}{s^2 - k^2}\,.$$

Remark:

As with derivatives we call the Laplace Transform an linear operator.

The Laplace transform of derivatives:

We have

$$\mathcal{L}\{f'(t)\}(s) \ = \int_0^\infty e^{-st} f'(t)dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt$$

$$= -f(0) + s\mathcal{L}\{f(t)\}(s) \ .$$

Repeating this procedure as often as necessary we get

$$\mathcal{L}\{f^{(n)}(t)\}(s) \ = -f^{n-1}(0) + s\mathcal{L}\{f^{(n-1)}(t)\}(s)$$

$$= -f^{n-1}(0) - sf^{n-2}(0) + s^2\mathcal{L}\{f^{(n-2)}(t)\}(s)$$

$$\begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

$$= s^n\mathcal{L}\{f(t)\}(s) \ - \sum_{k=1}^{n} \ s^{k-1} f^{(n-k)}(0)\,.$$

To find the inverse Lapalce transform we try to write the solution of the algebraic equation as a sum of functions for which we know from which function those are the transforms. Since the Laplace transfrorm is linear. The solution is the just the sum of this functions.

Laplace transform of trigonemetric and hypberboolic functions

$$\mathcal{L}\{\cos kt\}(s) \ = \frac{s}{k^2 + s^2\,.}$$

$$\mathcal{L}\{\sin kt\}(s) \ = \frac{k}{s^2 + k^2}\,.$$

$$\mathcal{L}\{\sinh kt\}(s) \; = \frac{k}{s^2 - k^2} \, .$$

$$\mathcal{L}\{\cosh kt\}(s) \; = \frac{s}{s^2 - k^2} \, .$$

Rules of finding Laplace Transforms and their inverse

1) Translations

$$\mathcal{L}\{e^{\alpha t} f(t)\}(s) \; = \mathcal{L}\{f(t)\}(s-a) \quad (= F(s-a))$$

and

$$e^{\alpha t} f(t) = \mathcal{L}^{-1}\{F(s-a)\}(t) \; .$$

2) Transforms of integrals:

$$\mathcal{L}\{\int_0^t f(\tau) d\tau\}(s) \; = \frac{1}{s}\mathcal{L}\{f(t)\}(s)$$

and

$$\mathcal{L}^{-1}\{\frac{1}{s}\mathcal{F}(s)\}(t) \; = \int_0^t f(\tau) d\tau \, .$$

Example:

$$\mathcal{L}\{f(t)\}(s) \; = \frac{1}{s(s-a)} \, , \text{ find } f \, .$$

We get

$$\mathcal{L}^{-1}(\frac{1}{s(s-a)}) = \int_0^t \mathcal{L}^{-1}(\frac{1}{(s-a)})(\tau) d\tau = \int_0^t e^{a\tau} d\tau$$

$$= \frac{1}{a}(e^{at} - 1) \, .$$

3) Laplace Transform of Convolutions

For two function f and g on the postive x-axis we define the convolution $f * g$ to be the function given by

$$f * g(t) = \int_0^t f(\tau) g(t-\tau) d\tau \, .$$

Firstly, we note that the convolution is commutative:

$$f * g(t) = g * f(t)\,.$$

The Laplace Transform of a convolution:

$$\mathcal{L}\{f(t) * g(t)\}\,(s) \; = \mathcal{L}\{f(t)\}\,(s)\; \mathcal{L}\{g(t)\}\,(s) \quad :$$

For the inverse Laplace Transform we gain the rule:

$$\mathcal{L}^{-1}\{\mathcal{F}(s)\mathcal{G}(s)\}\,(t) \; = \mathcal{L}^{-1}\{\mathcal{L}\{f\}\,(s)\; \mathcal{L}\{g\}\}\,(s)\;(t)$$

$$= f * g = \int_0^t f(\tau)g(t-\tau)d\tau\,.$$

4) Differentiation of the Laplace transform:

$$\mathcal{L}\{t^n f(t)\}\,(s) \; = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}\,(s)$$

and

$$\mathcal{L}^{-1}\{\frac{d^n}{ds^n}\mathcal{L}\{f(t)\}\,(s)\;\}\,(t) \; = (-1)^n t^n f(t)$$

5) Integration of the Laplace transform:

$$\mathcal{L}\{\frac{f(t)}{t}\}\,(s) \; = \int_s^\infty F(\sigma)d\sigma$$

Other formulas:

$$\mathcal{L}\{e^{at} f(t)\}\,(s) \; = \mathcal{L}\{f(t)\}(s-a)$$

$$\mathcal{L}^{-1}\{F(s-a)\}(t) = e^{at} f(t)$$

Lets consider for example

$$\begin{cases} u_{tt} - c^2\, u_{xx} = g(x,t) \\ u(0,t) = 0, \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

Viewing $u(x,t)$ as a function in t with parameters x we define

$$\mathcal{L}(u(x,t))(s) = \mathcal{U}(x,s) = \int_0^\infty u(x,t)e^{-st}\,dt.$$

and recall that

i) $\mathcal{L}(u_t(x,.)(s) = s\mathcal{U}(x,s) - u(x,0)$
$\mathcal{L}(u_{tt}(x,.))(s) = s^2\mathcal{U}(x,s) - s(u(x,0)) - u_t(x,0)\,.$

ii) $\mathcal{L}$ is a linear operation.

iii) The differentiation with respect to parameters interchanges with the transform.

$$\frac{\partial}{\partial x}\mathcal{L}(u(x,t))(s) = \mathcal{L}(u_x(x,t))s\,.$$

Hence taking the Laplace transform on both sides of the above PDE we get

$$s^2\mathcal{U}(x,s) - c^2\,\mathcal{U}_{xx}(x,s) = \mathcal{G}(x,s)$$

with

$$\mathcal{L}\{g\}(x,s) = \mathcal{G}(x,s).$$

For each s this is a second order ODE in x of the form

$$y''(x) - \frac{s^2}{c^2}y(x) = -\frac{1}{c^2}\mathcal{G}(x,s)\,.$$

Since we have that

$$v(x,s) = c_1(s)e^{-\frac{s}{c}x}$$

is a solution of the associated homogeneous equation, we can find a particular solution. On the half-line we get one initial condition for the ODE from the Laplace Transform of the boundary values. We also know that a Laplace Transform tends to zero as s tends to ∞.

Consequently we get from the ODE an unique function in s as the Laplace Transform of the solution. We then identify the solution using tables for (Inverse) Laplace transforms.

If g does not depend on x then the general solution of the ODE is given by

$$\mathcal{U}(x,s) = c_1(s)e^{-\frac{s}{c}x} + c_2(s)e^{\frac{s}{c}x} + \frac{1}{s^2}\mathcal{G}(s).$$

Taking the Laplace transform from the B.C.

$$\mathcal{L}(u(0,.))(s) = \mathcal{L}(0)(s) = 0$$

we get one initial condition for this ODE. Since the Laplace Transform tends to zero for large s the "constant" $c_2(s)$ needs to be zero and we can use the initial condition to determine $c_1(s)$. We get

$$0 = \mathcal{U}(0,s) = c_1(s)e^{-\frac{s}{c}x}|_{x=0} + \tfrac{1}{s^2}\mathcal{G}(s)$$

implying

$$c_1(s) = -\tfrac{1}{s^2}\mathcal{G}(s)$$

and hence

$$\mathcal{U}(x,s) = \tfrac{1}{s^2}(\mathcal{G}(s)(1 - e^{-\frac{s}{c}x})$$

is the Laplace transform of the solution, i.e.

$$u(x,t) = \mathcal{L}^{-1}(\mathcal{U}(x,.))(t).$$

Example:

$$g(t) = t^2 \quad \text{then } \mathcal{G}(s) = \tfrac{2}{s^3}$$

so

$$\mathcal{U}(x,s) = \tfrac{2}{s^5}(1 - e^{-\frac{s}{c}x})$$

and

$$u(x,t) = \tfrac{1}{12}(t^4 - (t - \tfrac{x}{c})^4 \text{ H } (t - \tfrac{x}{c}))$$

on account of a table or mathematica!

Parabolic equations and Fourier transforms.

Recall:
The Fourier transform of a function $f(x)$ is defined by

$$\mathcal{F}_f(\xi) = \int_{-\infty}^{+\infty} f(x)e^{i\xi x}\, dx$$

and we have

$$f(r) = \tfrac{1}{2\pi} \int_{-\infty}^{+\infty} \mathcal{F}_f(\xi)e^{-ir\xi}\, d\xi.$$

Note there are other definitions of the Fourier transform around, employing other constants. So the properties of the function my differ by some constants and/or multiples of i. The results for PDE however do not change.

Similar to the Laplace transform, for the Fourier transform differentiation changes to a multiplication with a certain factor, for appropriate functionns vanishing fast enough at infinity:

$$\mathcal{F}[f'](\xi) = \int_{-\infty}^{+\infty} f'(x)e^{+i\xi x}\, dx =$$

$$f(x)e^{+i\xi x}\Big|_{x\to-\infty}^{x\to+\infty} - i\xi \int_{-\infty}^{+\infty} f(x)e^{+i\xi x}\, dx$$

$$= (-i\xi)\mathcal{F}[f](\xi)$$

provided

$$0 = f(x)\cdot e^{+i\xi x}\Big|_{x\to-\infty}^{x\to+\infty} ,$$

which we always assume to hold when dealing with Fourier Transforms. So

$$\mathcal{F}[f'](\xi) = -i\xi\mathcal{F}[f]\,.$$

and

$$\mathcal{F}[f''](\xi) = -i\xi\ \ \mathcal{F}[f](\xi) = \ \ -\xi^2\hat{f}(\xi)\,.$$

We can repeat this calculation to find the higher order derivatives.

Properties of the Fourier transform:

1) Transform of the derivatives
   a) $\mathcal{F}[f'](\xi) = -i\xi\mathcal{F}[f]$
   b) $\mathcal{F}[f''](\xi) = -\xi^2\mathcal{F}[f]$
   c) $\mathcal{F}[f(^{(n)}](\xi) = (-i\xi)^n\mathcal{F}[f]$

2) Scaling:
   $\mathcal{F}[f(\alpha x)](\xi) = \frac{1}{\alpha}\mathcal{F}[f](\frac{\xi}{\alpha})$

3) Translation:
   $\mathcal{F}[f(a+x)](\xi) = e^{-ia\xi}\mathcal{F}[f](\xi)$

4) Multiplication by e^{-iax} :
   $\mathcal{F}[e^{iax}f(x)](\xi) = \mathcal{F}[f](\xi + a)$

5) Differentiation with respect to a parameter:

$$\mathcal{F}[\frac{\partial}{\partial s} f(s,x)](\xi) = \frac{\partial}{\partial s}\mathcal{F}[f(s,x)](\xi)\,.$$

6) The Fourier transform is a linear operation, i.e:

$$\mathcal{F}[\lambda f + g] = \lambda \mathcal{F}[f] + \mathcal{F}[g]\,.$$

7) Fourier integral formula

$$f(x) = \mathcal{F}^{-1}[\hat{f}(\xi)](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}[f](\xi) e^{-i\xi x} d\xi\,.$$

8) Fourier transform and convolution

For functions $f, g \in L^1(\mathbb{R})$ we define the convolution $f * g$ by

$$f * g(t) = \int_{-\infty}^{\infty} f(s) g(t-s) ds\,.$$

and we have

$$\mathcal{F}[u * v](\xi) = \mathcal{F}[u](\xi)\mathcal{F}[v](\xi), \quad \text{and}$$

$$(f * g)(x) = \mathcal{F}^{-1}[\mathcal{F}[u](\xi)\mathcal{F}[v](\xi)](x).$$

9) $\mathcal{F}[e^{-ax^2}]\xi = \sqrt{\frac{\pi}{a}} e^{-\xi^2/(4a)}\,.$

Indeed, note that

$$\frac{\partial}{\partial \xi}\mathcal{F}[e^{-ax^2}](\xi) = \frac{-i}{2a}\int_{-\infty}^{\infty} (-2ax) e^{-ax^2} e^{ix\xi} dx = \frac{-\xi}{2a}\mathcal{F}[e^{-ax^2}](\xi)\,,$$

integrating by parts. Hence the Fourier transforms of e^{-ax^2} satisfies the ODE

$$y' = \frac{-\xi}{2a} y\,,$$

with the initial condition

$$y(0) = \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

this gives 9).

<u>Example:</u>

Lets consider $u(x,t)$ satisfying

$$\begin{cases} u_t - k\, u_{xx} = 0 & (x,t) \in (-\infty,\infty) \times (0,\infty) \\ u(x,0) = f(x) \end{cases}$$

the Cauchy problem for the heat equation.

Dealing with t as an parameter, we define

$$\frac{\partial}{\partial s}\mathcal{F}[f(s,x)](\xi)\,.$$

and then obtain from the PDE:

$$\mathcal{U}_t(\xi,t)+k\xi^2\mathcal{U}(\xi,t)=0,$$

a first order ODE with parameter ξ which has the solution

$$\mathcal{U}(\xi,t)=C(\xi)e^{-k\xi^2 t}.$$

Now we determine $C(\xi)$ such that the initial condition are satisfied. We have

$$C(\xi)=\mathcal{U}(\xi,0)=(\mathcal{F}u(.,0))(\xi)=\hat{f}(\xi).$$

Hence the Fourier transform of a solution $u(x,t)$ is given by

$$\mathcal{U}(\xi,t)=\mathcal{F}[f](\xi)e^{-k\xi^2 t}\,.$$

a product of to functions.

In order to use the formula of the Fourier transform of a convolution, we would like to find g such that $\mathcal{F}[g]=e^{-\xi^2 kt}$. With $a=\dfrac{1}{4kt}$ in property 9) we get

$$\mathcal{F}[\tfrac{1}{\sqrt{4\pi kt}}e^{-x^2/4kt}](\xi)=\tfrac{1}{\sqrt{4\pi kt}}\mathcal{F}[e^{-x^2/4kt}](\xi)$$

$$=\tfrac{1}{\sqrt{4\pi kt}}\sqrt{4\pi kt}e^{\frac{-\xi^2 4kt}{4}}=e^{-\xi^2 kt}\,,$$

and we get from the formula of the Fourier transform of a convolution:

$$\mathcal{F}^{-1}[\mathcal{U}](x)=u(x,t)=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi kt}}e^{-(x-y)^2/4kt}f(y)\,dy$$

(provided f allows all the mathematical operations involved).

Note that for $t=0$ the solution $u(x,t)$ is not defined, so how can we say that it solves the initial value problem?

For a fixed, x let us consider the behavior of the solution as $t\to 0$, for bounded and continuous functions f, say, $|f(x)|\leq M$, for all x.

Substituting $p = \frac{y-x}{\sqrt{4kt}}$, then $\frac{dp}{dy} = \frac{1}{\sqrt{4kt}}$

and we get

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/4kt}\, dy = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-p^2}\, dp = 1$$

Recall: $\displaystyle\int_{-\infty}^{+\infty} e^{-p^2} dp = \sqrt{\pi}.$

So we can write

$$|u(x,t) - f(x)| \leq \int_{-\infty}^{x-\epsilon} \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/4kt} |f(y) - f(x)| dy$$

$$+ \int_{x-\epsilon}^{x+\epsilon} \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/4kt} |f(y) - f(x)| dy$$

$$+ \int_{x+\epsilon}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/4kt} |f(y) - f(x)|\, dy$$

$$\leq \frac{2M}{\sqrt{\pi}} \int_{-\infty}^{-\epsilon/\sqrt{4kt}} e^{-p^2} dp$$

$$+ \max_{\eta \in (x-\epsilon, x+\epsilon)} \{|f(x) - f(\eta)|\} \frac{1}{\sqrt{\pi}} \int_{-\epsilon/\sqrt{4kt}}^{\epsilon/\sqrt{4kt}} e^{-p^2} dp| \ ,$$

So for $t \to \infty$ we have that $|u(x,t) - f(x)|$ is bounded by

$$\max_{\eta \in (x-\epsilon, x+\epsilon)} \{|f(x) - f(\eta)|\} \text{ for all } \epsilon > 0\,.$$

Since f is continuous we must have $u(x,t) \to f(x)$ as $t \to 0$.

Example 1.

$$\begin{cases} u_t - k\, u_{xx} \\ u(x,0) \equiv 1. \end{cases}$$

So we have

$$u(x,t) = \tfrac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4kt}\, dy$$

Substituting again $p = \frac{y-x}{\sqrt{4kt}}$ we find

$$u(x,t) = \tfrac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-p^2}\, dp = 1$$

Note: $f \equiv 1$ does not allow all the operations leading to $(*)$ yet, the formula still represents the solution as can be checked directly.

Example 2.

$$\begin{cases} u_t - k\, u_{xx} = 0 \\ u(x,0) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \end{cases}$$

So

$$u(x,t) = \frac{1}{(4\pi kt)} \int_{0}^{\infty} e^{-(x-y)^2/4kt}\, dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{-x}{\sqrt{4kt}}}^{\infty} e^{-p^2}\, dp$$

$$= \frac{1}{\sqrt{\pi}} (\int_{0}^{\infty} e^{-p^2}\, dp \quad + \int_{\frac{-x}{\sqrt{4kt}}}^{0} e^{-p^2}\, dp$$

$$= \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{+x}{\sqrt{4kt}}} e^{-p^2}\, dp \;=\; \frac{1}{2}[1 + \ \text{Erf}\ (\frac{x}{\sqrt{4kt}})]\,.$$

Recalling the definiyion of the "error function"

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

Inhomogeneous heat equation

To solve this inhomogeneous equation

$$(1) \quad \begin{cases} u_t - k\, u_{xx} = f(x,t)\,, \\ u(x,0) = 0\,, \end{cases}$$

we use Duhamel's principle, again: i.e. we write

$$u(x,t) = \int_0^t V(x, t-\tau, \tau)\, d\tau\,.$$

where $V(x,t,\tau)$ is a solution of initial value problem.

$$(2) \quad \begin{cases} V_t - k\, V_{xx} = 0; \\ V(x,0,\tau) = f(x,\tau)\,. \end{cases}$$

We get

$$V(x,t,\tau) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{+\infty} e^{\frac{-(x-y)^2}{4kt}} f(y,\tau)\, dy$$

as the solution of (2). Hence the solution of (1) is

$$u(x,t) = \int_0^t \frac{1}{\sqrt{4\pi k(t-\tau)}} \int_{-\infty}^{+\infty} e^{\frac{-(x-y)^2}{4k(t-\tau)}} f(y,\tau)\, dy\, d\tau.$$

It is a good exercise to verify that indeed we found the solution of (1); for this we need

$$\left. \frac{1}{\sqrt{4\pi k(t-\tau)}} \int_{-\infty}^{+\infty} e^{\frac{-(x-y)^2}{4k(t-\tau)}} f(y,\tau)\, dy \right|_{t=\tau} = f(x,t)\,.$$

Strictly speaking the right hand side does not exist, but the equation can be justified as a limit as $t \to \tau$.

Heat equation on the half line

In order to deal with the heat equation of the half line we try to employ the reflection principle again, i.e., try to find a continuation to the whole line such that the result is satisfying the homogeneous B.C. We consider

$$\begin{cases} u_t - k\, u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u(0,t) = 0 \end{cases}$$

The solution for the Cauchy problem on the whole line was

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4kt} \tilde{\phi}(y)\, dy\,,$$

for some inital “value” $\tilde{\phi}$ defined on the whole line. At $x = 0$ this gives

$$u(0,t) = \frac{1}{4\pi kt} \int_{-\infty}^{+\infty} e^{-(y)^2/4kt} \tilde{\phi}(y)\, dy$$

We observe that the function $e^{-y^2/4kt}$ is even in y, hence if $\tilde{\phi}$ happens to be an odd function then we have $u(0,t) = 0$ for all t.
As a consequence we can now find a solution of (3) by setting

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} e^{\frac{-(x-y)^2}{4kt}} \tilde{\phi}(y)\, dy\,,$$

where $\tilde{\phi}(y)$ is an odd extension of ϕ to the whole line.
Note:
Since we are dealing with an integral over ϕ we do not need $\tilde{\phi}$ to be continuous at zero. In this case the solution will not be continuous at $(0,0)$.

If at $x = 0$ the BC is not homogeneous then again we can either use a change of the dependent variable by adding a function constant in x and solve the appropriate inhomogeneous equation, or use Laplace transform which often is easier:

Example:

$$\begin{cases} u_t - k\, u_{xx} = 0\,, \\ u(x,0) = 0\,, \\ u(0,t) = g(t)\,. \end{cases}$$

Taking the Laplace transform with respect to t gives

$$s\mathcal{U}(x,s) - u(x,0) - k\,\mathcal{U}_{xx}(x,s) = 0$$

providing an ODE for $\mathcal{U}(x,t)$ in x :

$$\mathcal{U}_{xx} - \frac{s}{k}\mathcal{U} = 0$$

which has the solution

$$\mathcal{U}(x,s) = C_1(s)e^{-\sqrt{\frac{s}{k}}x} + C_2(s)e^{\sqrt{\frac{s}{k}}x}\,.$$

Again we invoke that the Laplace transforms are usually decreasing functions in s, to conclude that the Laplace transform of the solution is of the form

$$\mathcal{U}(x,s) = C_1(s)e^{-\sqrt{\frac{s}{k}}x}\,.$$

To determine $C_1(s)$ we note that

$$\mathcal{U}(0,s) = C_1(s)\,,$$

and we get from the initial condition,

$$\mathcal{U}(0,s) = G(s) = \mathcal{L}\left\{g\right\}(s)\,.$$

Hence the Laplace transform of the solution is

$$\mathcal{U}(x,s) = G(s)\cdot e^{-\frac{x}{\sqrt{k}}\sqrt{s}}$$

We find the inverse Laplace transform of the second factor to be

$$\mathcal{L}^{-1}(e^{-\frac{x}{\sqrt{k}}\sqrt{s}})(t) = \frac{x}{\sqrt{4\pi t^3}}e^{-x^2/4kt}$$

and hence

$$\mathcal{L}^{-1}(G(s)\cdot e^{-\frac{x}{\sqrt{k}}\sqrt{s}}) = g(t) * (\frac{x}{\sqrt{4\pi t^3}}\,e^{\frac{-x^2}{4kt}})$$

where $(*)$ denotes the convolution of the two functions, i.e.,

$$u(x,t) = \int_0^t \frac{x}{\sqrt{4\pi(t-\tau)^3}}e^{-x^2/4k(t-\tau)}g(\tau)\,d\tau$$

which we can also write as

$$u(x,t) = \int_0^t \frac{x}{\sqrt{4\pi\tau^3}} e^{-x^2/4k\tau} g(t-\tau)\, d\tau.$$