

Section 2. Classification of Second Order PDEs

A second order linear PDE for an unknown $u = u(x, y)$ has the form

$$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + f u = g \quad (2.1)$$

so where a, b, c, d, e, f, g are constants or functions of x, y , but do not depend on u and/or its derivatives.

We use the following terminology

- 1) The $a u_{xx}, b u_{xy}$ and $c u_{yy}$ are called the highest or second order terms.
- 2) The other terms are called terms of lower order.
- 3) g is called the inhomogeneous term and we call the equation homogeneous if $g = 0$, otherwise the equation is called inhomogeneous

We want to find a characterization in terms of the coefficients of the second order derivatives a, b and c , which is independent of a change of variables.

Let

$$\xi = \alpha x + \beta y$$

$$\eta = \gamma x + \delta y$$

with

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0$$

Now with

$$u(x, y) = U(\xi, \eta) \Big|_{\substack{\xi = \alpha x + \beta y \\ \eta = \gamma x + \delta y}}$$

we get

$$u_x = \alpha U_\xi + \gamma U_\eta$$

$$u_y = \beta U_\xi + \delta U_\eta$$

and so

$$u_{xx} = \alpha^2 U_{\xi\xi} + 2\alpha\gamma U_{\xi\eta} + \gamma^2 U_{\eta\eta}$$

$$u_{xy} = \beta\alpha U_{\xi\xi} + (\delta\alpha + \beta\gamma) U_{\xi\eta} + \delta\gamma U_{\eta\eta}$$

$$u_{yy} = \beta^2 U_{\xi\xi} + 2\beta\delta U_{\xi\eta} + \delta^2 U_{\eta\eta}$$

Inserting these derivatives into the PDE gives:

$$\begin{aligned} a u_{xx} + b u_{xy} + c u_{yy} = \\ (a \alpha^2 + b\beta\alpha + c\beta^2)U_{\xi\xi} + \\ (2a \alpha\gamma + b(\delta\alpha + \beta\gamma) + 2c\beta\delta)U_{\xi\eta} + \\ (a\gamma^2 + b\delta\gamma + c\delta^2)U_{\eta\eta} = \\ \tilde{a}U_{\xi\xi} + \tilde{b}U_{\xi\eta} + \tilde{c}U_{\eta\eta}. \end{aligned}$$

With some algebra we find

$$\begin{aligned} \tilde{b}^2 - 4\tilde{a}\tilde{c} &= (b^2 - 4ac)(\alpha\delta - \gamma\beta)^2 = \\ (b^2 - 4ac) & \left(\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right)^2. \end{aligned}$$

So we find that under all possible transformation the sign of $D = (b^2 - 4ac)$ is invariant under a change of variables. We call the equation:

- i) hyperbolic, if $D > 0$;
- ii) parabolic, if $D = 0$;
- iii) elliptic, if $D < 0$.

Note that the type of the equation depends on the location in the plane if the coefficient a, b, c depend on x and y .

Examples

1) For

$$4 u_{xx} + 3 u_{xy} + u_x = 4 x$$

we get $D = 3^2$ hence this equation is hyperbolic.

2) For

$$x^2 u_{xx} + y u_{yy} = 0$$

we get $D = -4 x^2 \cdot y$

This PDE is

- i) hyperbolic for $y < 0$, and $x \neq 0$
- ii) elliptic for $y > 0$, and $x \neq 0$
- iii) parabolic for $y = 0$, or $x = 0$

Finding the canonical forms:

Hyperbolic equations ($D > 0$)

Choosing the transformation

$$\xi = x + \beta y \text{ and } \eta = x + \delta y$$

with

$$\beta = \frac{-b + \sqrt{D}}{2c} \quad \delta = \frac{-b - \sqrt{D}}{2c}$$

we get from (2.1) equation

$$U_{\xi\eta} + \text{lower order terms} = 0$$

for

$$U(\xi, \eta) \Big|_{\substack{\xi=\alpha x+\beta y \\ \eta=\gamma x+\delta y}} \cdot$$

In case $c = 0$ interchange the role of x and y .

In the homework you will be asked to find a transformation such that (2.1) implies.

$$\tilde{U}_{\xi\xi} - \tilde{U}_{\eta\eta} + \text{lower order terms} = 0.$$

Parabolic equations ($D = 0$):

The transformation

$$\xi = x, \quad \text{and} \quad \eta = x - \frac{b}{2c}y$$

gives first

$$aU_{\xi\xi} + \text{lower order terms} = 0,$$

and multiplying by $\frac{1}{a}$, we get

$$U_{\xi\xi} + \text{lower order terms} = 0.$$

(In case $c = 0$, we must have that $b = 0$, that is the equation is already in the desired form.)

Elliptic equations ($D < 0$) .

We may assume that $a > 0$. (otherwise we multiply the equation by (-1), this does not change D) and we note that

$$D < 0, a > 0 \Rightarrow c > 0 \text{ and } 2a > b\sqrt{\frac{a}{c}}.$$

Hence we can define the transformation

$$\begin{cases} \xi = x + \frac{-b}{2c} y \\ \eta = \frac{-\sqrt{-D}}{2c} y. \end{cases}$$

(2.1) now implies

$$\frac{-D}{4c}(U_{\xi\xi} + U_{\eta\eta}) + \text{lower order terms} = 0,$$

and multiplying by $\frac{-D}{4c}$, we get

$$U_{\xi\xi} + U_{\eta\eta} + \text{lower order terms} = 0.$$

Note that we can write

$$U(\xi, \eta) \Big|_{\begin{pmatrix} \xi \\ \eta \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}} = u(x, y)$$

with

$$A = \begin{pmatrix} 1 & \frac{-b}{2c} \\ 0 & \frac{-\sqrt{-D}}{2c} \end{pmatrix}.$$

It should be noted that there are other transformations providing a canonical form of the equations. In particular as you are asked to show in the homework the Laplace equation is invariant under rigid transformations.

Examples

$$\begin{array}{ll} \alpha) & u_{xx} + u_{yy} = 0, & \text{is called Laplace equation} \\ & u_{xx} + u_{yy} = f(x, y), & \text{is called Poisson equation} \end{array}$$

$$\beta) \quad u_t - k u_{xx} = 0,$$

is called the heat equation (or diffusion equation)

$$\gamma) \quad u_{tt} - c^2 u_{xx} = 0,$$

is called the wave equation, where the constant c is referred to as wave speed.

Let us consider

$$u_{tt} - c^2 u_{xx} = 0.$$

Introducing the transformation (change of variables)

$$\begin{cases} \xi = x + ct \\ \eta = x - ct \end{cases}$$

we get

$$u_x = U_\xi + U_\eta$$

$$u_{xx} = U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\xi}$$

$$u_t = c(U_\xi - U_\eta)$$

$$u_{tt} = c^2(U_{\xi\xi} - 2U_{\eta\xi} + U_{\eta\eta})$$

$$0 = u_{tt} - c^2 u_{xx} = -4c^2 U_{\xi\eta}$$

Hence we obtain the equivalent equation

$$U_{\eta\xi} = 0$$

writing

$$0 = U_{\eta\xi} = \frac{\partial}{\partial \xi} U_\eta$$

we conclude that U_η is a constant function in ξ , so we get

$$U_\eta = f(\eta)$$

If \mathcal{F} is a anti-derivative of f we conclude that

$$U(\xi, \eta) = \mathcal{F}(\eta) + \mathcal{G}(\xi)$$

where the \mathcal{G} is constant with respect to η .

Hence

$$U(\xi, \eta) = \mathcal{F}(\eta) + \mathcal{G}(\xi)$$

is a solution of $U(\xi, \eta)$ for all possible choices of ("sufficiently smooth") functions \mathcal{F} and \mathcal{G} and we obtain

$$u(x, t) = \mathcal{F}(x - ct) + \mathcal{G}(x + ct).$$

Other examples of finding the general solution explicitly:

$$U_{\xi\eta} + U_\xi = f(\eta).$$

Auxiliary Conditions

Depending on the domain D in the plane in which the PDE is considered, the auxiliary condition vary greatly. As a rule of thumb for a well posed problem we need as many auxiliary conditions as we have derivatives. For functions of two variables, that would be four in case of the hyperbolic equation and elliptic equation and three for the parabolic equation. This is in particular valid for problems in bounded space domains. In case of unbounded domains some times fewer conditions are necessary. The most important types of conditions are the initial conditions and boundary conditions.

- 1) The initial condition usually prescribe the value of the function and its time derivative at a given time (mostly for $t = 0$,)
- 2) If $\partial\Omega$ is the boundary of a domain $\Omega \subset \mathbb{R}^n$ and n is the outer normal along the boundary then the most important boundary conditions are
 - i) $u|_{\partial\Omega} = f$ Dirichlet condition
 - ii) $\frac{\partial}{\partial n}u|_{\partial\Omega} = f$ Neumann condition
 - iii) $au + b\frac{\partial}{\partial n}u|_{\partial\Omega} = f$ Robin condition

Typical situations are the following.

For the hyperbolic problems

- 1) D the x, t -plane:

$$(x, t) \in (-\infty, +\infty) \times (-\infty, +\infty)$$

$$(IVP) \begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{(hyp. PDE in D)} \\ u(x, 0) = f(x) & \text{Initial condition for } u, \\ u_t(x, 0) = g(x) & \text{Initial condition for } u_t \end{cases}$$

is called the initial value problem for the wave equation.

- 2) D parts of the x, t -plane such as

- i) $D = (a, +\infty) \times (\infty, +\infty)$

$$(IBVP) \begin{cases} u_{tt} - c^2 u_{xx} = 0, & \text{(hyp. PDE in D)} \\ u(x, 0) = f(x) & \text{Initial condition for } u, \\ u_t(x, 0) = g(x) & \text{Initial condition for } u_t \\ u(a, t) = h(t) & \text{Boundary condition} \\ & \text{at } x = a \end{cases}$$

and

ii) $D = (a, b) \times (-\infty, +\infty)$

$$(IBVP) \begin{cases} u_u - c^2 u_{xx} = 0 & \text{(hyp. PDE in D)} \\ u(x, 0) = f(x) & \text{Initial condition for } u, \\ u_t(x, 0) = g(x) & \text{Initial condition for } u_t \\ u(a, t) = h(t) & \text{Boundary condition} \\ & \text{at } x = a \\ u(b, t) = g(t) & \text{Boundary condition} \\ & \text{at } x = b \end{cases}$$

Both above are called initial boundary value problems (IBVP).

Parabolic problems.

1) D the x, t -half-plane: $t > 0$.):

$(x, t) \in (-\infty, +\infty) \times (0, +\infty)$

$$(IVP) \begin{cases} u_t - k u_{xx} = 0 & \text{(parab. PDE in D)} \\ u(x, 0) = f(x) & \text{Initial condition for } u, \end{cases}$$

is called the initial boundary value problem for the heat equation.

2) D parts of the upper half of the x, t -plane such as

i) $D = (a, +\infty) \times (0, +\infty)$

$$(IBVP) \begin{cases} u_u - k u_{xx} = 0 & \text{(parab. PDE in D)} \\ u(x, 0) = f(x) & \text{Initial condition for } u, \\ u(a, t) = h(t) & \text{Boundary condition} \\ & \text{at } x = a \end{cases}$$

and

ii) $D = (a, b) \times (0, +\infty)$

$$(IBVP) \begin{cases} u_t - k u_{xx} = 0 & \text{(parab. PDE in D)} \\ u(x, 0) = f(x) & \text{Initial condition for } u, \\ u(0, t) = h(t) & \text{Boundary condition} \\ & \text{at } x = 0 \\ u(b, t) = g(t) & \text{Boundary condition} \\ & \text{at } x = b \end{cases}$$

Both above are called initial boundary value problems (IBVP).

Elliptic problems

Elliptic equations with D a connected open set in the plane, with "smooth" boundary ∂D

$$(D.P) \begin{cases} u_{xx} + u_{yy} = 0 & \text{Laplace equation} \\ u|_{\partial D} = g(x, y) & \text{Dirichlet condition .} \end{cases}$$

is called a Dirichlet problem (or boundary value problem of the first kind)

$$(N.P) \begin{cases} u_{xx} + u_{yy} = f & \text{Poisson equation} \\ \frac{\partial}{\partial n} u|_{\partial D} = g(x, y) & \text{Neumann condition .} \end{cases}$$

is called a Neumann problem (for the Poisson equation, or BVP of the second kind)

$$(R.P) \begin{cases} u_{xx} + u_{yy} = 0 & \text{Laplace equation} \\ au + b \frac{\partial}{\partial n} u|_{\partial D} = g(x, y) & \text{Robin condition} \end{cases}$$

is called Robin problem or boundary problem of the third kind.

Example

The initial value problem of the wave equation.

$$u_{tt} - c^2 u_{xx} = 0$$

$$u(x, 0) = f(x)$$

$$u_t(x, 0) = g(x)$$

We want to find F and G of the general solution

$$u(x, t) = F(x - ct) + G(x + ct).$$

Hence we must have with

$$u_t(x, t) = -cF'(x - ct) + cG'(x + ct) :$$

$$f(x) = u(x, 0) = F(x) + G(x)$$

$$\begin{aligned}
g(x) = u_t(x, 0) &= -c F'(x) + c G'(x) \\
&= c(G - F)'(x)
\end{aligned}$$

Hence

$$(G - F)(x) - \frac{1}{c} \int_0^x g(t) dt + C$$

or

$$\frac{1}{c} \int_0^x g(t) dt + C = -F(x) + G(x).$$

$$2 \quad G(x) = f(x) + \frac{1}{c} \int_0^x g(t) dt + C$$

$$2 \quad F(x) = f(x) - \frac{1}{c} \int_0^x g(t) dt - C$$

and so

$$\begin{aligned}
u(x, t) &= \frac{1}{2} f(x - ct) + f(x + ct) + \\
&\quad \frac{1}{2c} \left[\int_0^{x+ct} g(r) dr - \int_0^{x-ct} g(r) dr \right]
\end{aligned}$$

This we can further simplify observing that

$$\int_{x-ct}^0 g(r) dr = - \int_0^{x-ct} g(r) dr$$

and we get d'Alembert's solution of the wave equation:

$$u(x, t) = \frac{1}{2} f(x - ct) + \frac{1}{2} f(x + ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(r) dr$$

Diffusive and dispersive equations

Functions of the form

$$u(x, t) = \phi(x - ct), \quad x \in \mathbb{R}$$

are often called traveling waves. More generally for $x \in \mathbb{R}^n$ function of the form

$$u(x, t) = \phi(v \cdot x - \omega t),$$

are called plane waves, (with wavefront normal to $v \in \mathbb{R}^n$ and speed

$$c = \frac{|\omega|}{|v|}.)$$

In particular for plane waves of the form

$$u(x, t) = Ae^{i(kx - \omega t)} \quad x \in \mathbb{R},$$

the following terminology is used:

- i) k is called the wave number
- ii) ω is called the time (temporal) frequency.
- iii) A is called the amplitude.

Substituting $u = e^{i(kx - \omega t)}$ into a PDE provides the so-called dispersion relation

$$\omega = \omega(k),$$

in order u to be a solution.

The properties of the dispersion relation are used for some classification of those PDEs:

The equation is called dispersive, if $\omega = \omega(k)$ is real valued function and nonlinear.

The equation is called diffusive (or dissipative), if $\omega = \omega(k)$ is complex valued.

Note, if $\omega(k) = \alpha(k) + i\beta(k)$, then $i(kx - \omega t) = \beta(k)t + i(kx - \alpha(k)t)$ and so

$$u(x, t) = e^{\beta(k)t} e^{i(k \cdot x - \alpha(k)t)}.$$

That is, the amplitude $e^{\beta(k)t}$ either grows exponentially or vanishes (dissipates) exponentially.

If, however, the frequency $\omega(k)$ is real, then the exponent of u has no real part so the solution consist of two waves:

$$u(x, t) = \cos(kx - \omega t) + i \sin(kx - \omega t),$$

with speed

$$c = \frac{|\omega|}{|k|}.$$

If ω is a linear function of k then the speed of the wave is independent of the frequency. If however, ω depends nonlinear on k then the speed of the wave depends on the frequency. This phenomenon is called dispersion.

Summarizing we have:

i) The equation is called dispersive, if $\omega(k)$ is real for real values k , and $\omega''(k) \neq 0$.

ii) The equation is called diffusive, if $\omega(k)$ is complex for real values k .

Of course there are equation which are neither dispersive nor diffusive, such as the wave equation.

Example:

For the PDE

$$u_t - u_{xxx} = 0,$$

we get the dispersive equation $\omega = k^3$.

The function $\omega(k) = k^3$ is real for real values of k and $\omega''(k) \neq 0$, hence we call this equation dispersive.