PDE (Math 4163)　　　　Spring 2016

Some historical notes.

PDE arose in the context of the development of models in the physics of continuous media, e.g. vibrating strings, elasticity, the Newtonian gravitational field of extended matter, electrostatics, fluid flows, and later by the theories of heat conduction, electricity and magnetism. In addition, problems in differential geometry gave rise to nonlinear PDE's such as the Monge˘ Ampeȷ re equation and the minimal surface equations. The classical calculus of variations in the form of the Euler-Lagrange principle gave rise to PDE's and the Hamilton-Jacobi theory, which had arisen in mechanics, stimulated the analysis of first order PDE's. During the 18th century, the foundations of the theory of a single first order PDE and its reduction to a system of ODE's was carried through in a reasonably mature form. The classical PDE's which serve as paradigms for the later development also appeared first in the 18th and early 19th century. The one dimensional wave equation

$$u_{tt} - c^2 u_{xx} = 0\,,$$

was introduced and analyzed by d'Alembert in 1752 as a model of a vibrating string. His work was extended by Euler (1759) and later by D. Bernoulli (1762) to 2 and 3 dimensional wave equations

$$u_{tt} - c^2 \Delta u = 0\,,$$

in the study of acoustic waves.

The Laplace equation

$$\Delta u = 0\,,$$

was first studied by Laplace in his work on gravitational potential fields around 1780. The heat equation

$$u_t - c^2 \Delta u = 0\,,$$

was introduced by Fourier in his celebrated memoir "The orie analytique de la chaleur" (1810-1822). Thus, the three major examples of second-order PDE's hyperbolic, elliptic and parabolic had been introduced by the first decade of the 19th century, though their central role in the classification of PDE's, and related boundary value problems, were not clearly formulated until later in the century. Besides the three classical examples, a profusion of equations, associated with major physical phenomena, appeared in the period between 1750 and 1900:

The Euler equation of incompressible fluid flows, 1755.

The minimal surface equation by Lagrange in 1760 (the first major application of the Euler-Lagrange principle in PDE's).

The Monge-Ampere equation by Monge in 1775.

The Laplace and Poisson equations, as applied to electric and magnetic problems, starting with Poisson in 1813, the book by Green in 1828 and Gauss in 1839.

The Navier Stokes equations for fluid flows in 1822-1827 by Navier, followed by Poisson (1831) and Stokes (1845).

Linear elasticity, Navier (1821) and Cauchy (1822).

Maxwell's equation in electromagnetic theory in 1864.

The Helmholtz equation and the eigenvalue problem for the Laplace operator in connection with acoustics in 1860.

The Plateau problem (in the 1840's) as a model for soap bubbles.

The Korteweg - De Vries equation (1896) as a model for solitary water waves.

(Adapted from

Haim Brezis, Felix Browder,

Partial Differential Equations in the 20th Century

Advances in Mathematics 135, 76-144 (1998).

Conservation laws on the real line

We begin the discussion of PDEs showing how some of theses can be obtained from basic physical conservation laws.

We consider a substance or other physical quantities like electrical charge distributed over an interval (a, b) of the real line, during time interval (q, s), say, with the "space" variable denoted by x, and the "time" variable denoted by t,

In order to be able to describe the situation mathematically, we make the following basic assumptions.

(A_1) Density:

There is a (smooth) function $u(x,t)$ defined on $(a,b) \times (q,s)$, such that

$$U_{\alpha,\beta}(t) = \int_\alpha^\beta u(x,t)dx$$

is the amount of the quantity at time t in the interval (α,β) for all α,β. The function u is called the density of the quantity.

(A_2) Flux:

There is a smooth function $\phi(x,t)$ defined on $(a,b) \times (q,s)$, such that

$$\Phi_{\sigma,\tau}(x) = \int_\sigma^\tau \phi(x,t)dt$$

is the amount of the quantity passing (in the direction of the) space variable through "a point" $x \in (a,b)$ for any time interval (σ,τ); the function ϕ is called the flux of the quantity

(A_3) Source (or Sink):

There is a function $f(x,t)$ on $(a,b) \times (q,s)$, such that

$$G_{\alpha,\beta,\sigma,\tau} = \int_\sigma^\tau \int_\alpha^\beta f(x,t)\ dxdt$$

the gain (or loss) of the quantity in $(\tau,\sigma) \times (\alpha,\beta)$; the function f is called the source function.

Now our experience relates those three quantities in the following manner: The change

$$U_{\alpha,\beta}(\tau) - U_{\alpha,\beta}(\sigma)$$

of the quantity in an interval (α,β) from a time $t=\sigma$ to a time $t=\tau$ equals

$$\Phi_{\sigma,\tau}(\alpha) - \Phi_{\sigma,\tau}(\beta)$$

the amount passed into (α,β) through $x=\alpha$, minus the amount passed out of the interval at $x=\beta$,

plus the amount $G_{\alpha,\beta,\sigma,\tau}$ generated on (α,β) in the time interval (σ,τ) i.e.:

$$U_{\alpha,\beta}(\tau) - U_{\alpha,\beta}(\sigma) =$$

$$\Phi_{\sigma,\tau}(\alpha) - \Phi_{\sigma,\tau}(\beta) + G_{\alpha,\beta,\sigma,\tau}\,.$$

Consequently, for all $\alpha, \beta, \sigma, \tau$ as above we have:

$$\begin{gathered}\int_{\alpha}^{\beta} u(x,\tau)dx - \int_{\alpha}^{\beta} u(x,\sigma)dx = \\ \int_{\sigma}^{\tau} \phi(\alpha,t)dt - \int_{\sigma}^{\tau} \phi(\beta,t)dt + \int_{\sigma}^{\tau}\int_{a}^{\beta} f(x,t)\end{gathered} \tag{1.1}$$

Definition 1.1
(1.1) is called the (integral form) of the “conservation law”
$u(x,t)$ is called the density (distribution) of the quantity,
$\phi(x,t)$ is called the flux of the quantity, and
$f(x,t)$ is called the source function.

Because of the Fundamental Theorem of Calculus we can rewrite (1.1):

$$\int_{\alpha}^{\beta}\int_{\sigma}^{\tau} u_t(x,t)dt\ dx + \int_{\sigma}^{\tau}\int_{\alpha}^{\beta} \phi_x(x,t)dx\ dt =$$

$$\int_{\sigma}^{\tau}\int_{\alpha}^{\beta} f(x,t)dx\ dt$$

or interchanging the integration on the first term we get

$$\int_{\sigma}^{\tau}\int_{\alpha}^{\beta} u_t(x,t) + \phi_x(x,t)dxdt = \int_{\sigma}^{\tau}\int_{\alpha}^{\beta} f(x,t)dxdt$$

Since this is valid for any two pairs of number (α,β), (σ,τ) such that $(\alpha,\beta)\times(\sigma,\tau) \subset (a,b)\times(q,s)$, we get (from a not quite trivial theorem of Calculus) the

Differential form of the conservation laws:

$$u_t(x,t) + \phi_x(x,t) = 0, \tag{1.2}$$

if no gain or loss occurs or

$$u_t(x,t) + \phi_x(x,t) = f(x,t) \tag{1.3}$$

if there is a gain or a loss.

Remarks:

1) These "laws" (1.1)-(1.3) are not proven in a mathematical sense, but consequence from our basic assumption in order to use mathematics to describe "real life" phenomena.
2) This law provides only one equation for the two unknown quantities u and ϕ, so we need further information providing a relation between flux and the density to determine how the quantity evolves in time. Those may vary considerably with the particular situation under consideration. Usually, the resulting relations are called constitutive relations, often provided in the form $\phi(x.t) = g(x, t, u, u_x, ...)$.

Example:

Assume we have observed that a density function $u(x, t)$ moves along a line with a constant speed c, say, a signal along a wire. That is we have

$$u(x + ch, t + h) = u(x, t)\,,$$

for $(x, t) \in (-\infty, \infty) \times (-\infty, \infty)$, and h in $\mathbb{R}$.

On the other hand, we must have that

$$\int_{\sigma}^{\sigma+h} \phi(\alpha, t)dt\,,$$

the amount of the quantity which passed through $x = \alpha$ in the time interval $(\sigma, \sigma + h)$ equals

$$\int_{\alpha-ch}^{\alpha} u(x, \sigma)dx\,,$$

the amount of the quantity in the interval $(\alpha - ch, \alpha)$ at time σ. So we get

$$\int_{\sigma}^{\sigma+h} \phi(\alpha, t)dt = \int_{\alpha-ch}^{\alpha} u(x, \sigma)dx \tag{1.5}$$

If ϕ and u are continuous, then the mean value theorem provides

$$\int_{\sigma}^{\sigma+h} \phi(\alpha, t)dt = h \cdot \phi(\alpha, \overline{s})$$

for some $\overline{s}$, with $\sigma < \overline{s} < \sigma + h$. and

$$\int_{\alpha-ch}^{\alpha} u(x, \sigma)dx = ch \cdot u(\xi, \sigma)$$

for some ξ, with $\alpha - ch < \xi < \alpha$, Hence dividing (1.5) by h and letting h go to zero we find

$$cu(x,t) = \phi(x,t)\,.$$

That is: the density is proportional to the flux, and for all $(x,t) \in (-\infty,\ \infty) \times (-\infty, \infty)$ we get from (1.2):

$$u_t + cu_x = 0.$$

Suppose now that we have obtained the information that for a certain quantity with density function $u(x,t)$ on $((-\infty, \infty) \times (-\infty, \infty)$ say) the flux is proportional to the density function. Can we conclude that the density function is moving with constant speed?

In mathematical terms that is: Does a solution of

$$u_t(x,t) + cu_x(x,t) = 0.$$

has the form

$$u(x,t) = u(x + ch, t + h)\,?$$

Remark:
There are, of course, other constitutive relations. Such as $\phi = au_x$ i.e. the flux is proportional to the slope of the distribution, as for the heat and diffusion equations. We will be concerned with those later.

1. Linear First order PDE's.

Definition:
The equation.

$$au_x + bu_y + du = g\,,$$

for R $= (x,y) \in (\alpha, \beta) \times (\gamma, \delta)\,,$

with a, b, d, g (continuous) functions on R is called a linear first order PDE on R.
If $g \equiv 0$ then the equation is called homogeneous.

Equations of the form

$$au_x + bu_y = 0\,, \tag{1.6}$$

Recall: Directional derivatives:

In calculus we have defined

$$f_{\underline{v}}(\underline{w}) = (\frac{\partial}{\partial \underline{v}} f)(\underline{w}) = \lim_{h\to 0} \frac{1}{h}(f(\underline{w} + h\underline{v}) - f(\underline{w}))$$

to be the derivative of f at $\underline{w}$ in the direction of $\underline{v}$. Here f is a real valued function defined, say, on D an open subset $\mathbb{R}^n$, $\underline{w} \in D$ and $\underline{v}$ is any vector in $\mathbb{R}^n$.

Examples:

The partial derivatives are directional derivatives in the direction of the basis vectors of the standard basis in $\mathbb{R}^n$, i.e.:

$$f_{\underline{e}_k} = \frac{\partial}{\partial \underline{e}_k} f = \frac{\partial}{\partial x_k} f = \; f_{x_k}$$

In general we have

$$f_{\underline{v}} = \underline{v} \cdot \nabla f = \sum_{i=1}^{n} \; v_i \frac{\partial}{\partial x_i} f$$

for functions

$$f = f(x_1, \ldots, x_n) \text{ and vectors } \underline{v} = (v_1, \ldots, v_n)\,.$$

In our case we get from (1.6) for $\underline{v} = (a, b)$ and $u = u(x, y)$ that

$$0 = au_x + bu_y = \underline{v} \cdot \nabla u = u_{\underline{v}}$$

i..e.: if we are moving along a line with direction vector v then the derivative of u in this direction is zero, hence there is no change along this line. In other words $u(x, y)$ is constant along lines of direction $\underline{v}$.

Next we recall that any point $\underline{r} = (x, y)$ on a line L satisfies the equation

$$\underline{r} = \underline{r}_0 + t\underline{v}$$

for some fixed point $\underline{r}_0 = (x_0, y_0)$ on that line and with direction vector $\underline{v} = (a, b)$.

Since $\underline{n} \cdot \underline{v} = 0$ for $n = (b, -a)$ we find:

$$0 = \underline{n} \cdot (\underline{r} - \underline{r}_0) = bx - ay - b \cdot x_0 + ay_0\,,$$

or

$$bx - ay = c\,, \tag{1.7}$$

for some constant $c = bx_0 - ay_0$.

The above observation includes that for each real number c we have a line given by (1.7) along which the solution u is constant, say $f(c)$, i.e., the solution u defines a function $f : c \to f(c)$ and we have

$$u(x, y) = f(c)\,, \quad \text{if} \ \ (bx - ay) = c.$$

Hence a solution of (1.6) (if there is one) has to be of the form,

$$u(x, y) = f(bx - ay)\,.$$

On the other hand, if f is differentiable we get

$$u_x(x, y) = \frac{d}{dx}(f(bx - ay)) = f'(bx - ay)b$$

and

$$u_y(x, y) = \frac{d}{dy}(f(bx - ay)) = -f'(bx - ay)a$$

and so

$$(au_x - bu_y)(x, y) = (ab - ba)(f'(bx - ay)) = 0\,.$$

We found indeed a solutions of (1.6) .

Initial value problems:

With the fact that $u(x, y) = f(bx - ay)$ we have used all the information provided by the PDE. We have to infer auxiliary condition to determined the solution uniquely. The most important one is the so called initial condition:

Suppose that $u(x, y)$ satisfies the PDE above and we know the function of u is $g(x)$ at a certain line, $y = 0$, say. Then we must solve

$$(IVP) \qquad \begin{cases} au_x + bu_y = 0 \\ u(x, 0) = g(x) \end{cases}$$

This is called an initial value problem (IVP).

We know, the solutions must be of the form

$$u(x, y) = f(bx - ay)$$

implying

$$g(x) = u(x, 0) = f(bx - 0) = f(bx)\,.$$

We know the function g by assumption and want to determine f.
Setting $r = bx$ we solve for x and get

$$x = \frac{r}{b}.$$

That is, the function f must be given by

$$f(r) = g(\frac{1}{b}r),$$

writing r for the independent variable. Hence for the solution we get

$$u(x,y) = f(bx - ay) = g(\frac{1}{b}(bx - ay)),$$

So

$$u(x,y) = g(x - \frac{a}{b}y),$$

is a solution of the (IVP).
As shown earlier it solves the PDE and we have

$$u(x,0) = g(x).$$

Change of variables

A slightly different approach is to introduce "characteristic" variables:

First we consider equation of the form

$$a(x,y)u_x + b(x,y)u_y = 0$$

We introduce the new coordinate system

$$\begin{cases} \xi = s(x,y) \\ \eta = y \end{cases}$$

where $s(x,y) = c$ is an implicit solution of the ODE

$$y'(x) = \frac{b(x,y)}{a(x,y)},$$

or more generally an implicit solution of the ODE

$$a(x,y)dy = b(x,y)dx.$$

That later equation showing clearly that it is equivalent to think of those curves as functions in x or functions in y.

To motivate that choice of coordinate system, we consider $u(x,y)$ along a graph given by the solution $y(x)$ of the above ODE. Then we have

$$\frac{d}{dx}u(x,y(x)) = u_x + u_y y' = 0\,,$$

and so

$$a(x,y)u_x + b(x,y)u_y = 0$$

Hence the solution does not change along the integral curves of the ODE. The curves defined by the equations $s(x,y) = c$ are called the characteristics of the PDE.
Note, differentiating $s(x,y(x)) = c$, with respect to x we get

$$as_x + bs_y = 0\,, \quad \text{or} \quad \frac{s_x(x,y)}{s_y(x,y)} = -\frac{b(x,y)}{a(x,y)}\,.$$

For the second variable η we can chose any linear combination of x and y, as long as that is not colinear with ξ in an open subset of the plane.

Now, with the above change of variables, we get a function $\mathcal{U} = \mathcal{U}(\xi,\eta)$ by setting

$$\mathcal{U}(\xi,\eta)\Big|_{\substack{\xi=s(x,y)\\ \eta=y}} = u(x,y)\,.$$

So $u(x,y) = \mathcal{U}(s(x,y),y)$

and we have

$$u_x = \mathcal{U}_\xi s_x \quad \text{and} \quad u_y = \mathcal{U}_\xi s_y + \mathcal{U}_\eta\,.$$

From the PDE we obtian the equation

$$0 = a\mathcal{U}_\xi s_x + b(\mathcal{U}_\xi s_y + \mathcal{U}_\eta) = (as_x + bs_y)\mathcal{U}_\xi + b\mathcal{U}_\eta = b\mathcal{U}_\eta\,.$$

or

$$\mathcal{U}_\eta = 0\,.$$

That is $\mathcal{U}$ has to be constant in η, so it can vary only in ξ, and we must have

$$\mathcal{U}(\xi,\eta) = G(\xi)$$

for some function G, providing

$$u(x,y) = G(s(x,y))\,.$$

Again G has to be determined by auxiliary conditions.

Example:

Consider

$$u_x + x u_y = 0\,.$$

The ODE for the characteristic curves is given by

$$y'(x) = x.$$

which has the solution $y = \dfrac{x^2}{2} + c$ or

$$y - \frac{x^2}{2} = c\,.$$

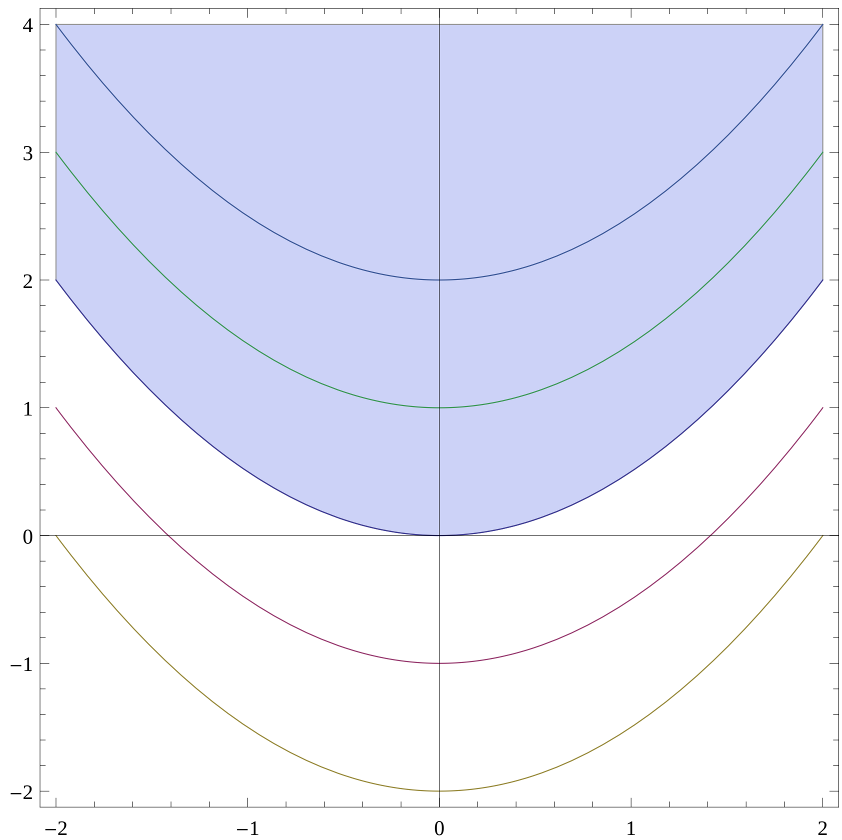

Fig. 2 : Some characteristic lines of the PDE

As the new coordinates we get

$$\begin{cases} \xi = y - \dfrac{x^2}{2}\,, \\ \eta = y \end{cases}$$

So introducing the function $\mathcal{U} = \mathcal{U}(\xi, \eta)$ such that

$$\mathcal{U}(\xi, \eta)\Big|_{\substack{\xi = y - \frac{x^2}{2} \\ \eta = y}} = u(x, y)\,,$$

we get

$$u_x = -\mathcal{U}_\xi\, x \quad \text{and} \quad u_y = \mathcal{U}_\xi + \mathcal{U}_\eta\,;$$

Because of the PDE we must have

$$\mathcal{U}_\eta = 0 \ \text{ i.e: } \mathcal{U}(\xi, \eta) = G(\xi)$$

Consequently the general solution of the PDE is

$$u(x, y) = G(y - \frac{x^2}{2})\,.$$

Now consider the auxiliary condition

$$u(x, 0) = g(x)\,.$$

It provides

$$g(x) = u(x, 0) = G(-\frac{x^2}{2})\,.$$

To find G as a function of a real variable, r say, we set $r = -\dfrac{x^2}{2}$ and solve for x. This gives

$$x = \sqrt{-2r}\,.$$

So the function $G(r) = g(\sqrt{-2r})$ and the solution of the initial value problem is

$$u(x, y) = g(\sqrt{-2(y - \frac{x^2}{2})}\,) = g(\sqrt{x^2 - 2y})\,.$$

Note that $u(x, y) = g(\sqrt{x^2 - 2y})$ is defined for all (x, y) for which $2y \le x^2$. Since for negative x we have $x = -\sqrt{x^2}$ we need $g(x) = g(-x)$ in order for that initial value problem to have a solution at all.

Also, for the region in the plane for which $2y > x^2$, that is the area (shaded in Fig. 2) above the parabola through the origin, the solution is not determined by g at all.

Apparently the auxiliary condition cannot be chosen freely on just any curve. Note, the line given by $y = 0$ is parallel to the tangent of the characteristic through the origin at that very point.

We say the problem is "well posed" if the auxiliary condition(s) provide the existence of a unique solution in the domain in which we consider the PDE. The above example indicates that possibly problems are not well posed if the auxiliary conditions are given on curves which are characteristic at some points i.e.: the auxiliary condition are chosen on curves which are tangent to a characteristic curve at some points.

The line given by $x = 0$ is nowhere tangent to characteristics and we obtain a unique solution for

$$(*) \qquad \begin{cases} u_x + xu_y = 0 \\ u(0,y) = \phi(y) \end{cases}.$$

Indeed we must have

$$\phi(y) = u(0,y) = G(y - \frac{x^2}{2})|_{x=0} = G(y)\,.$$

So the solution has to be of the form $u(x,t) = \phi(y - \frac{x^2}{2})$ and it is straight forward to check that it is indeed a solution of $(*)$.

Exmaple 2
For constant coefficients,

$$au_x + bu_y = 0\,,$$

the ODE for the characterstics is

$$y' = \frac{b}{a}\,,$$

which has the solutions $y = \frac{b}{a}x + d$, foe some constant d. Since a is constant we get equivalently

$$ay - bx = c\,, \text{ with } c = da\,.$$

Setting

$$\begin{cases} \xi = ay - bx\,, \\ \eta = y \end{cases}$$

we define the function $\mathcal{U}$ by

$$\mathcal{U}(\xi,\eta)\Big|_{\substack{\xi=ay-bx \\ \eta=y}} = u(x,y)$$

i.e.:

$$u(x,y) = \mathcal{U}(ay - bx, y)$$

Note: that in the new variables $\mathcal{U}$ does not vary if we keep ξ, constant. So no matter would linear combination of x and y we choose for the second variable η, such that ξ and η are linearly independent,the solution does not change if ξ is fixed and η changes. We always will have $\mathcal{U}_\eta = 0$.

Indeed we get

$$u_x = -b\mathcal{U}_\xi \quad \text{and} \quad u_y = a\mathcal{U}_\xi \; + \mathcal{U}_\eta\,;$$

or

$$0 = au_x + bu_y = -ab\mathcal{U}_\xi + ba\mathcal{U}_\xi + b\mathcal{U}_\eta$$

and for $b \neq 0$, (otherwise ξ and η would be linearly independent.

$$\mathcal{U}_\eta = 0\,.$$

Hence

$$\mathcal{U} = C(\xi) \quad \text{i.e.: } u(x,y) = C(ay - bx)\,.$$

For $u = u(x,t)$ and the equation

$$u_t + cu_x = 0\,,$$

we have $a = c$ and $b = 1$ so $\xi = t - \dfrac{x}{c}, \eta = t$ and the general solution

$$u(x,t) = C(ct - x) = \tilde{C}(x - ct)\,.$$

Back to our problem: does the solution of

$$u_t + cu_x = 0\,,$$

model a fixed shape moving along a line with constant speed?

As a solution of the PDE $u(x,t)$ must be of the form $u(x,t) = f(x-ct)$. Does this describe a density moving to the right? Let's see:

$$u(x+ch, t+h) = f(x+ch-c(t+h)) =$$

$$f(x-ct) = u(x,t).$$

Yes, it does.

Note: The argument of u is a point, (vector) in $\mathbb{R}^2$, where as the argument of f is a real number!

Example:

$$g(x) = \begin{cases} \cos x \ , & \frac{\pi}{2} \leq x \leq \frac{\pi}{2} \ ; \\ 0 \ , & \text{otherwise} \ ; \end{cases}$$

then

$$u(x,t) = \begin{cases} \cos(x-ct) \ , & \frac{\pi}{2} \leq x-ct \leq \frac{\pi}{2} \ ; \\ 0 \ , & \text{otherwise} \ . \end{cases}$$

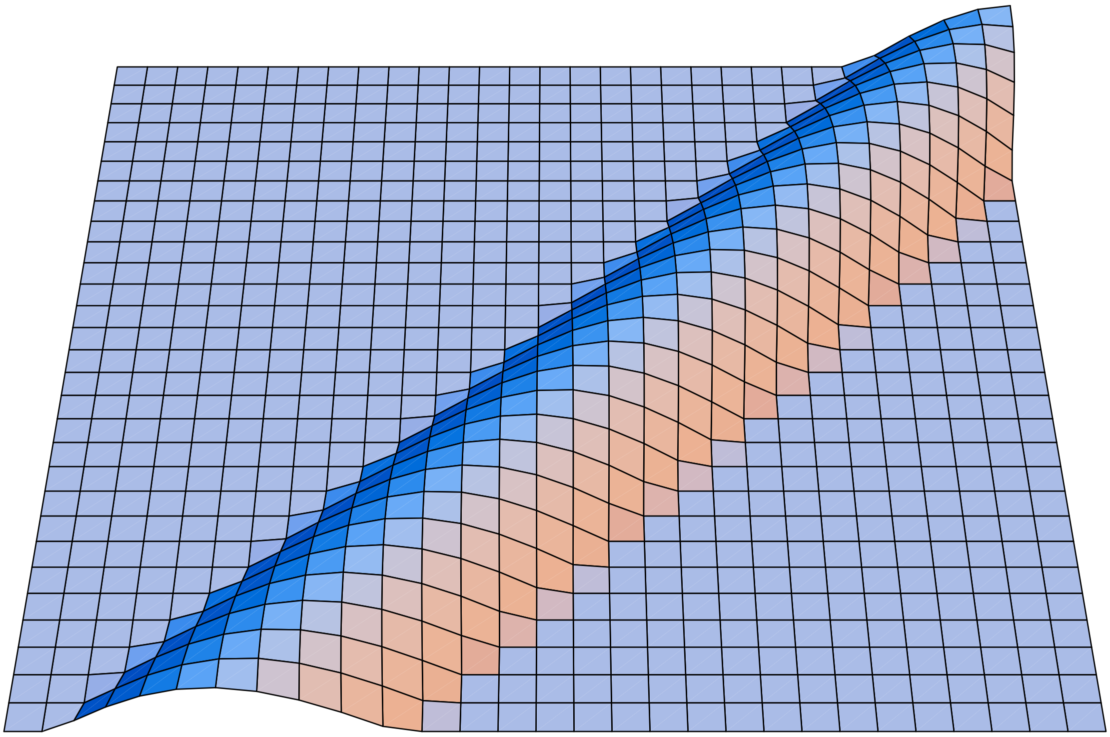

Fig. 1 : Solution of the IPV

There are many curves along which we can prescribe the values for u, for instance consider

$$(IVP) \qquad \begin{cases} u_t + cu_x = 0 \\ u(0,t) = \phi(t) \end{cases}$$

then we must have

$$\phi(t) = u(0,t) = f(-ct)\,.$$

With $\tau = -ct$ we get $t = -\frac{\tau}{c}$, and $f(\tau) = \phi(-\frac{\tau}{c})$, providing the solution,

$$u(x,t) = \phi(t - \frac{x}{c})\,.$$

1.2 Linear equation with non-constant coefficients:

We consider equation of the form

$$a(x,y)u_x + b(x,y)u_y + d(x,y)u = f(x,y)$$

and again introduce the new coordinate system

$$\begin{cases} \xi = s(x,y) \\ \eta = y \end{cases}$$

where $s(x,y) = c$ is a implicit solution of the ODE.

$$y'(x) = \frac{b(x,y)}{a(x,y)}\,,$$

For $\mathcal{U} = \mathcal{U}(\xi,\eta)$

$$\mathcal{U}(\xi,\eta)\Big|_{\substack{\xi=s(x,y)\\ \eta=y}} = u(x,y)\,.$$

i.e.: $u(x,y) = \mathcal{U}(s(x,y),y)$

we have

$$u_x = \mathcal{U}_\xi s_x \quad \text{and} \quad u_y = \mathcal{U}_\xi s_y \; + \mathcal{U}_\eta\,.$$

and obtain from the PDE the equation

$$b(x,y)\mathcal{U}_\eta(\xi,\eta))\Big|_{\substack{\xi=s(x,y)\\ \eta=y}} + d(x,y)u = f(x,y)\,.$$

With $\mathcal{D}(\xi,\eta)$ and $\mathcal{F}(\xi,\eta)$ given by

$$\mathcal{D}(\xi,\eta)\Big|_{\substack{\xi=s(x,y)\\ \eta=y}} = \frac{d(x,y)}{b(x,y)} \quad \text{and} \quad \mathcal{F}(\xi,\eta)\Big|_{\substack{\xi=s(x,y)\\ \eta=y}} = \frac{f(x,y)}{b(x,y)}\,,$$

respectively, we find that $\mathcal{U}$ is subject to the linear ODE

$$\mathcal{U}_\eta + \mathcal{D}\mathcal{U} = \mathcal{F}\,,$$

in η, (for fixed ξ.) For fixed ξ the (general) solution are determined up to a constant C, which may depend on ξ, i.e.: the solution is of the form

$$\mathcal{U}(\xi,\eta) = w(\eta,\xi) + C(\xi)h(\eta,\xi)\,.$$

and we determine $C(\xi)$ using the auxiliary condition, where $w(\eta,\xi)$ is one solution of the inhomogeneous ODE

$$\mathcal{U}_\eta + \mathcal{D}(\xi,\eta)\mathcal{U} = \mathcal{F}\,(\xi,\eta)\,,$$

and $h(\eta,\xi)$ is a solution of the homogeneous ODE

$$\mathcal{U}_\eta + \mathcal{D}(\xi,\eta)\mathcal{U} = 0\,,$$

in η for fixed ξ. The dependence of h,w and C on ξ is then determined with the help of th auxiliary condition.
Recall: The general formular for the solution of a linear first order ODE

$$y' + P(x)y = Q(x)\,,$$

is given by

$$y = e^{-\int P(x)dx}(\int Q(x)e^{\int P(x)dx}dx + C)\,;$$

Also recall that the integrating factor is given by

$$\rho(x) = \exp(\int P(x))dx\,.$$

Example 1

For the general initial boundary value problem with constant coefficients

$$\begin{cases} au_x + bu_y + du = g,\,, \\ u(x,0) = h(x)\,. \end{cases},$$

we get

$$y' = \frac{b}{a}\,,$$

so $s(x,y) = ay - bx$, and we have to solve the ODE

$$\mathcal{U}_\eta + \frac{d}{b}\mathcal{U} = \frac{g}{b}\,,$$

which has the soltion

$$\mathcal{U} = C(\xi)e^{-\frac{d}{b}\eta} + \frac{g}{d}\,,$$

and we get the general solution

$$u(x,y) = C(ay - bx)e^{-\frac{d}{b}y} + \frac{g}{d}\,.$$

The initial condition provides

$$h(x) = u(x,0) = C(-bx) + + \frac{g}{d}\,.$$

With $r = -bx$, and $x = -\frac{r}{b}$, we find

$$C(r) = h(-\frac{r}{b}) - \frac{g}{d}\,,$$

which yields the solution

$$u(x,y) = (h(x - \frac{a}{b}y) - \frac{g}{d})e^{-\frac{d}{b}y} + \frac{g}{d}\,.$$

Example 2:

$$\begin{cases} xu_x + yu_y + yu = x\,, \\ u(x,1) = g(x)\,. \end{cases}$$

First lets find the general solution of the PDE:

The ODE for the characteristics is given by

$$y'(x) = \frac{y}{x.}$$

which is a separable ODE with the solution $y = \tilde{c}x$ or

$$\tilde{c} = \frac{y}{x} \text{ or equivalently } c = \frac{x}{y}.$$

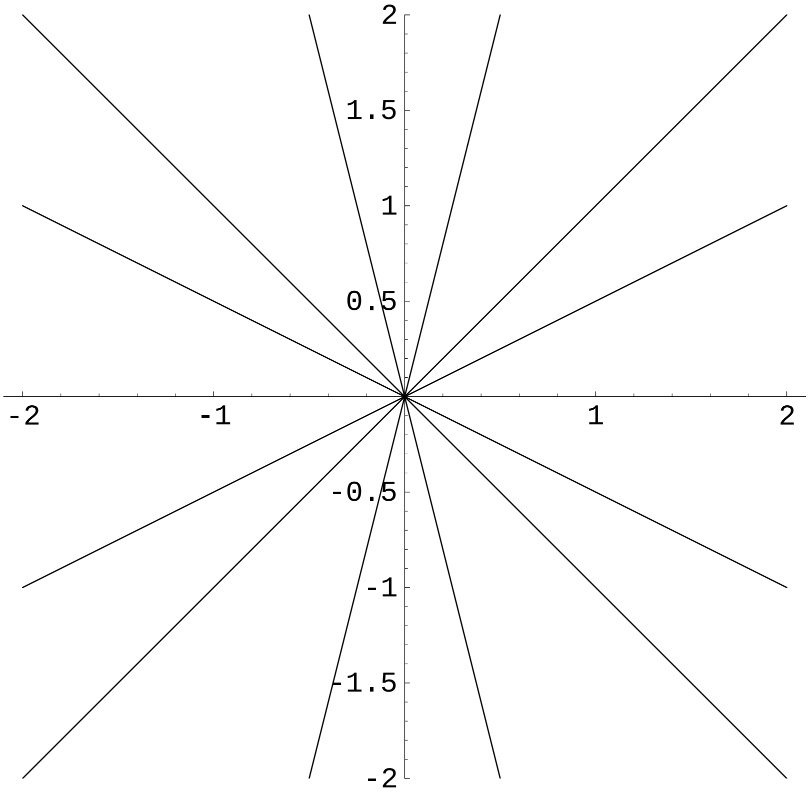

Fig. 3 : Some characteristic lines of the PDE

As the new coordinates we get

$$\begin{cases} \xi = \dfrac{x}{y}\,, \\ \eta = y \end{cases}$$

So for

$$\mathcal{U}(\xi,\eta)\Big|_{\substack{\xi=\frac{x}{y}\\ \eta=y}} = u(x,y)$$

We get

$$u_x = \mathcal{U}_\xi\,\frac{1}{y} \quad \text{and} \quad u_y = -\mathcal{U}_\xi\,\frac{x}{y^2} + \mathcal{U}_\eta\,;$$

Because of the PDE we must have

$$y\mathcal{U}_\eta\Big|_{\substack{\xi=\frac{x}{y}\\ \eta=y}} + yu = x\,,$$

or

$$\mathcal{U}_\eta + \mathcal{U} = \xi \,,$$

which has the solution:

$$\mathcal{U}(\xi, \eta) = \xi + C(\xi)e^{-\eta} \,.$$

Consequently the general solution of the PDE is

$$u(x, y) = (\frac{x}{y}) + C(\frac{x}{y})e^{-y} \,.$$

The initial condition provides

$$g(x) = u(x, 1) = x + C(x)e^{-1} \,.$$

With $r = x$, we get

$$C(r) = e \cdot (g(r) - r) \,,$$

and so the solution of the problem is

$$u(x, y) = \frac{x}{y} + e^{1-y}(g(\frac{1}{x}) - \frac{x}{y}) \,.$$

Example 3:

$$\begin{cases} u_x + y^2 u_y + u = x + \dfrac{1}{y} \,, \\ u(x, 0) = g(x)(?) \,. \end{cases}$$

First lets find the general solution of

$$u_x + y^2 u_y + u = x + \frac{1}{y} \,.$$

The ODE for the characteristics is given by

$$y'(x) = y^2.$$

which is a separable ODE with the solution $-\dfrac{1}{y} = x + c$ or $y = \dfrac{1}{c - x}$.

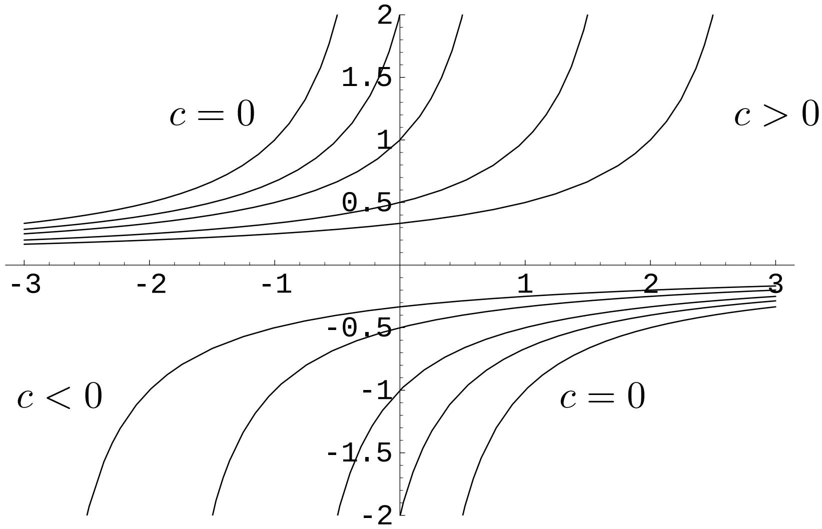

Fig. 4 : Some characteristic lines of the PDE

Note that the sketch shows, that the initial value problem is ill posed.
As the new coordinates we get

$$\begin{cases} \xi = x + \dfrac{1}{y}\,, \\ \eta = y \end{cases}$$

So for

$$\mathcal{U}(\xi,\eta)\Big|_{\substack{\xi = x+\frac{1}{y} \\ \eta = y}} = u(x,y)$$

We get

$$u_x = \mathcal{U}_\xi \quad \text{and} \quad u_y = -\mathcal{U}_\xi \frac{1}{y^2} + \mathcal{U}_\eta\,.$$

Because of the PDE and $x = \xi - \dfrac{1}{\eta}$ we must have

$$\mathcal{U}_\eta + \frac{1}{\eta^2}\mathcal{U} = \frac{1}{\eta^2}\xi\,.$$

That is a linear first oder ODE which has the solution

$$\mathcal{U} = \xi + C(\xi)e^{1/\eta}\,.$$

(Integrating factor $(\exp(\int p d\eta)\,, \quad p = \dfrac{1}{\eta^2}\,.\ )$
Consequently the general solution of the PDE is

$$u(x,y) = x + \frac{1}{y} + C(x + \frac{1}{y})e^{1/y}\,. \qquad (*)$$

Since the problem as intended is ill posed, let us consider two other auxiliary conditions $u(x,1) = g(x)$, and $u(0,y) = f(y)$.

For $u(x,1) = g(x)$, we have

$$g(x) = u(x,1) = x + 1 + C(x+1)e\,,$$

With $r = x + 1$ and $x = r - 1$ this implies

$$C(r) = \frac{-r + g(r-1)}{e}\,.$$

and so inserting the argument $x + \frac{1}{y}$ in C we get the soltion

$$u(x,y) = x + \frac{1}{y} + (-(x + \frac{1}{y}) + g(x + \frac{1}{y} - 1))e^{\frac{1}{y}-1}$$

The second auxiliary condition $f(y) = u(0,y)$ provides

$$f(y) = u(0,y) = \frac{1}{y} + C(\frac{1}{y})e^{1/y}\,,$$

With $r = \frac{1}{y}$ and $y = \frac{1}{r}$ this implies

$$C(r) = \frac{-r + f(\frac{1}{r})}{e^r}\,.$$

Again we have to replace r by the argument of C in (*) that is $x + \frac{1}{y}$, and we get

$$u(x,y) = x + \frac{1}{y} + (f(\frac{y}{xy+1}) - x - \frac{1}{y})e^{-x}\,.$$