CONVEXITY II SPRING 2000
TEST 2 – ANSWERS

In order to get full credit, all answers must be accompanied by appropriate justifications.

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1. Let $P_1, \ldots, P_k$ be polytopes in $\mathbb{E}^n$ and let $\lambda_1, \ldots, \lambda_k$ be non-negative real numbers. Prove that each of the following sets is a polytope:

a). $\text{conv}(P_1 \cup \cdots \cup P_k)$;
b). $\bigcap_{i=1}^k P_i$;
c). $\lambda_1 P_1 + \cdots + \lambda_k P_k$.

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a). If we let V denote the set of all vertices of all the polytopes $P_1, \ldots, P_k$ then V is a finite set. Clearly $\bigcup_{i=1}^k P_i \subset \text{conv}V$ and so $\text{conv}\bigcup_{i=1}^k P_i \subset \text{conv}V$. Conversely, $V \subset \bigcup_{i=1}^k P_i$ and so $\text{conv}\bigcup_{i=1}^k P_i = \text{conv}V$ which gives the desired result.

b). Each P_i is a bounded polyhedral set and so their intersection immediately inherits this property. It follows that the intersection is also a polytope (see Theorems 20.7 and 20.9).

c). If $\lambda \geqslant 0$ and $P = \text{conv}(v_1, \ldots v_t)$ then $x \in \lambda P \Leftrightarrow$ there are the usual $\lambda_1, \ldots, \lambda_t$ such that $x = \lambda(\lambda_1 v_1 + \cdots + \lambda_t v_t) \Leftrightarrow x \in \text{conv}(\lambda v_1, \ldots, \lambda v_t)$ and so λP is a polytope. Let $P = \text{conv}(v_1, \ldots, v_t)$ and $Q = \text{conv}(w_1, \ldots, w_t)$ be polytopes; we can assume the same number of v_i as w_i by repetition, if necessary. We have $x \in P + Q \Leftrightarrow$ there are the usual $\lambda_1, \ldots, \lambda_t$ and $\mu_1, \ldots, \mu_t$ such that

$$x = \lambda_1 v_1 + \cdots + \lambda_t v_t + \mu_1 w_1 + \cdots + \mu_t w_t = \sum_{i,j=1}^{t} \lambda_i \mu_j (v_i + w_j).$$

This is equivalent to $x \in \text{conv}(v_1, \ldots, v_t, w_1, \ldots, w_t)$ and so $P + Q$ is a polytope. The required result now follows by induction.

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2. Denote by $\{p, q\}$ the regular 3-polytope all of whose facets are p-gons and for which there are q facets at each vertex. Let P be a 4-polytope all of whose facets are congruent to $\{p, q\}$ and for which each edge lies on precisely r such facets. Prove that the numbers $f_0(P)$, $f_1(P)$, $f_2(P)$, $f_3(P)$ are proportional (the same ratio in each case) to the numbers

$$\frac{1}{q} + \frac{1}{r} - \frac{1}{2}, \quad \frac{1}{r}, \quad \frac{1}{p}, \quad \frac{1}{p} + \frac{1}{q} - \frac{1}{2}.$$

If ϕ is the angle between two faces of $\{p, q\}$ that meet at an edge, show that $r\phi < 2\pi$. Use the fact that

$$\phi = 2\sin^{-1}\left(\cos\frac{\pi}{q} \Big/ \sin\frac{\pi}{p}\right)$$

to deduce that there at most six possibilites for the ordered triple $\{p,q,r\}$.

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Let $F,\ E,\ V$ be the numbers of faces, edges and vertices of $\{p,q\}$. Each 2-face of P lies on exactly 2 facets and therefore $2f_2(P) = Ff_3(P)$. We also have $Ef_3(P) = rf_1(P)$. Euler's formula gives

$$f_0(P) = \left(\frac{E}{r} - \frac{F}{2} + 1\right) f_3(P).$$

Using the values of $F,\ E,\ V$ in terms of $p,\ q$ gives

$$\frac{f_0(P)}{f_1(P)} = 1 + \frac{r}{q} - \frac{r}{2} \qquad \frac{f_0(P)}{f_2(P)} = \frac{p}{r} + \frac{p}{q} - \frac{p}{2} \qquad \frac{f_3(P)}{f_2(P)} = \frac{p}{q} - \frac{p}{2} + 1.$$

This yields the desired result.

If e is an edge of P, we can choose a hypeplane H such that $H \cap P = e$. We let E be a hyperplane orthogonal to e which meets e. Then $H \cap E$ is a 2-dimensional plane in E. We let P_1 be the polytope $E \cap P$ and note that $H \cap E$ supports P_1 at the vertex $E \cap e$. Each facet of P, which contains e, intersects E to yield a facet of P_1. The angle between each pair of facets of P_1 containing $e \cap E$ is ϕ. There are r of these facets and so $r\phi < 2\pi$.

We are told that $(\sin\phi/2)(\sin\pi/p) = \cos\pi/q$, and so $\cos\pi/q < (\sin\pi/r)(\sin\pi/p)$. The only possible pairs for $\{p,q\}$ are $\{3,3\},\ \{3,4\},\ \{3,5\},\ \{4,3\},\ \{5,3\}$. This quickly leads to the following six possibilities for $\{p,q,r\}$,

$$\{3,3,3\},\ \{3,3,4\},\ \{3,3,5\},\ \{3,4,3\}, \{4,3,3\},\ \{5,3,3\}.$$

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3. Let rB be a ball in $\mathbb{E}^2$ which is circumscribed to a regular k-gon P. Find the smallest $\delta > 1$ such that $P \subset rB \subset \delta P$.

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Let θ denote the angle subtended by an edge of P at the center of the ball. Then $\theta = 2\pi/k$. Elementary trigonometry now shows that the distance from the midpoint of the edge to the center is $r\cos\pi/k$. This must be the radius of the insphere of P. It follows that $\delta = 1/(\cos\pi/k)$.

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4. Let C be a cube in $\mathbb{E}^3$ whose circumsphere is the unit ball B. Describe C^* as fully as you can. Prove that if S is an n-simplex with $o \in \mathrm{int}S$, then so is S^*.

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If L is the edge length of the cube then we have $3L^2 = 4$ (Pythagoras' theorem) and so $L = 2/\sqrt{3}$. It follows that $C = \bigcap_{i=1}^{3}\{x \in \mathbb{E}^3 : |\langle x, e_i\rangle| \leqslant 1/\sqrt{3}\}$. Consequently $C^* = \mathrm{conv}(\pm e_1\sqrt{3}, \pm e_2\sqrt{3}, \pm e_3\sqrt{3})$. This is the regular octahedron with the indicated vertices.

Let $v_1, \ldots, v_{n+1}$ be the (affinely independent) vertices of S. For $i = 1, \ldots, n+1$, let H_i be the affine hull of this set of vertices with v_i removed. The affine independence shows that H_i is a hyperplane. Since $o \in \mathrm{int}S$ we can choose vectors $w_1, \ldots, w_{n+1}$ such that $H_i = \{x \in \mathbb{E}^3 : \langle x, w_i\rangle = 1\}$ and $S = \bigcap_{i=1}^{n+1}\{x \in \mathbb{E}^3 : \langle x, w_i\rangle \leqslant 1\}$. It follows that $S^* = \mathrm{conv}(w_1, \ldots, w_{n+1})$. This gives the desired result, since we already know from the boundedness of S that $o \in \mathrm{int}S^*$.

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5. Let K be a non-empty set in $\mathbb{E}^n$. Prove that $K^{**} = \mathrm{cl}\,\mathrm{conv}(K \cup \{o\})$.

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We always have $K \subset K^{**}$ and K^{**} is a closed convex set containing o. Thus $\mathrm{cl}\,\mathrm{conv}(K \cup \{o\}) \subset K^{**}$. Furthermore $\mathrm{cl}\,\mathrm{conv}(K \cup \{o\})$ is a closed convex set containing K. It follows that $K^{**} \subset (\mathrm{cl}\,\mathrm{conv}(K \cup \{o\}))^{**} = \mathrm{cl}\,\mathrm{conv}(K \cup \{o\})$, as required.