CONVEXITY II SPRING 2000
TEST 1

In order to get full credit, all answers must be accompanied by appropriate justifications.

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1. Let $K_1, \dots, K_4$ be convex bodies in $\mathbb{E}^n$. Prove that

$$D(\operatorname{conv}(K_1 \cup K_2), \operatorname{conv}(K_3 \cup K_4)) \leqslant D(K_1, K_3) + D(K_2, K_4).$$

where D denotes the Hausdorff metric.

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Choose $\rho > 0$ so that $K_1 \subset K_3 + \rho B$ and $K_2 \subset K_4 + \rho B$. If $z \in \operatorname{conv}(K_1 \cup K_2)$ there are $x \in K_1,\ y \in K_2$ and $0 \leqslant \lambda \leqslant 1$ such that $z = \lambda x + (1-\lambda)y$ (perhaps this should be proved since it requires the convexity of $K_1,\ K_2$ and is not true in general). If $x \in K_1$ there is a $v_1 \in \rho B$ and a $k_3 \in K_3$ such that $x = k_3 + v_1$. If $y \in K_2$ there is a $v_2 \in \rho B$ and a $k_4 \in K_4$ such that $y = k_4 + v_2$. Now $z = \lambda k_3 + (1-\lambda)k_4 + \lambda v_1 + (1-\lambda)v_2 \in \operatorname{conv}(K_3 \cup K_4) + \rho B$. Consequently $\operatorname{conv}(K_1 \cup K_2) \subset \operatorname{conv}(K_3 \cup K_4) + \rho B$. Similarly, if $K_3 \subset K_1 + \rho B$ and $K_4 \subset K_2 + \rho B$ we can deduce that $\operatorname{conv}(K_3 \cup K_4) \subset \operatorname{conv}(K_1 \cup K_2) + \rho B$. The combination of these two observations shows that

$$\begin{aligned} D(\operatorname{conv}(K_1 \cup K_2), \operatorname{conv}(K_3 \cup K_4)) &\leqslant \max\{D(K_1, K_3), D(K_2, K_4)\} \\ &\leqslant D(K_1, K_3) + D(K_2, K_4). \end{aligned}$$

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2. Assume that the convex bodies $K_1, K_2, \dots,$ in $\mathbb{E}^n$, converge to K with $\dim K = n$. Let $p \in \operatorname{int} K$. For each $i = 1, 2, \dots$ let $\alpha_i = \sup\{\alpha \geqslant 0 : \alpha(K_i - p) + p \subset K\}$ and put $D_i = \alpha_i(K_i - p) + p$. Prove that $\alpha_i \to 1$ and $D_i \to K$ as $i \to \infty$.

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We may choose $r > 0$ so that $2rB + p \subset K$ and note that $h_K(u) > \langle p, u\rangle + r$ for all unit vectors u. Furthermore, we know that $h_{K_i} \to h_K$ uniformly on the set of unit vectors as $i \to \infty$. It follows that we may assume $h_{K_i}(u) > \langle p, u\rangle + r$ for all unit vectors u, or, equivalently, $K_i \supset rB + p$ for all i. For convenience, we put $f_i = h_{K_i} - \langle p, \cdot\rangle$ for $i = 1, 2, \dots$ and $f = h_K - \langle p, \cdot\rangle$. Clearly, $f_i \to f$ uniformly on the set of all unit vectors as $i \to \infty$, and $f(u) > r,\ f_i(u) > r$ for all i and all unit vectors u. It follows that $f_i/f \to 1$ uniformly on the set of all unit vectors as $i \to \infty$. Note that $\alpha_i = \inf\{f(u)/f_i(u) : \|u\| = 1\}$. It follows that $\alpha_i \to 1$ as

$i \to \infty$. To finish, note that $h_{D_i} = \alpha_i f_i + \langle p, \cdot\rangle \to f + \langle p, \cdot\rangle = h_K$ uniformly on the set of unit vectors as $i \to \infty$. Thus $D_i \to K$ as $i \to \infty$.

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3. Let $K, K_1, K_2, \ldots$ be convex bodies in $\mathbb{E}^n$. Prove that $K_i \to K$ as $i \to \infty$ if and only if the two following conditions are true:

a). each point of K is the limit of a sequence $(x_i)_{i=1}^{\infty}$ with $x_i \in K_i$, for each $i = 1, 2, \ldots$;

b). the limit of any convergent sequence $(x_{i_j})_{j=1}^{\infty}$ with $x_{i_j} \in K_{i_j}$, for each $j = 1, 2, \ldots$ is in K.

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First, assume that $K_i \to K$ as $i \to \infty$. Let $x \in K$ and put $x_i = p(K_i, x)$ for $i = 1, 2, \ldots$. Then $x_i \in K_i$ and $d(x, x_i) = d(\{x\}, K_i) \leqslant D(K, K_i)$. Hence $x_i \to x$, and so **a)** is true. If $x_{i_j} \to x$ as $j \to \infty$ and $x \notin K$, there is an $r > 0$ such that $(rB + x) \cap (K + rB) = \emptyset$. This is impossible, since, if j is sufficiently large, then $d(x_{i_j}, x) < r$ and $x_{i_j} \in K_{i_j} \subset K + rB$.

Conversely, assume **a)** and **b)** both hold and let ϵ be given. We will show that

$$K \subset K_i + \epsilon B \qquad \text{if } i \text{ is sufficiently large} \qquad (1)$$
$$K_i \subset K + \epsilon B \qquad \text{if } i \text{ is sufficiently large} \qquad (2)$$

If (1) were false, there would be a sequence $\left(x_{i_j}\right)_{j=1}^{\infty}$ with $x_{i_j} \in K$ and $d(x_{i_j}, K_{i_j}) \geqslant \epsilon$ for each j. The compactness of K allows us to assume that $x_{i_j} \to x \in K$ as $j \to \infty$. By **a)** there is a sequence $y_{i_j} \in K_{i_j}$ with $y_{i_j} \to x$ as $j \to \infty$. But then $d(x_{i_j}, y_{i_j}) \to 0$ as $j \to \infty$, which contradicts the fact that $d(x_{i_j}, K_{i_j}) \geqslant \epsilon$ for each j. So we have proved that (1) is true. If (2) were false there would be a sequence $\left(x_{i_j}\right)_{j=1}^{\infty}$ with $x_{i_j} \in K_{i_j}$ and $d(x_{i_j}, K) \geqslant \epsilon$ for each $j = 1, 2, \ldots$. We can use **a)** to choose $y_{i_j} \in K_{i_j} \cap (K + \epsilon B)$ for each $j = 1, 2, \ldots$. The convexity of K_{i_j} therefore allows us to assume that $d(x_{i_j}, K) = \epsilon$ for $j = 1, 2, \ldots$ and therefore that the sequence converges. According to **b)** the limit must be in K, but this contradicts the fact that $d(x_{i_j}, K) = \epsilon$ for $j = 1, 2, \ldots$.

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4. In $\mathbb{E}^3$, let L_1 be the closed line segment joining the origin to the point $(1, 0, 0)$, let L_2 be the closed line segment joining the origin to $(0, 1, 0)$ and let L_3 be the closed line segment joining the origin to $(1, 0, 1)$. Let P be the vector sum of these three line segment. For $\rho \geqslant 0$ we put $P_\rho = P + \rho B$, where B is the closed unit ball in $\mathbb{E}^3$. Express the volume of P_ρ as a polynomial in ρ.

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P is a paralleloptope with four edges parallel to each of the line segments L_1, L_2, L_3. If $x \in P_\rho$ there are four disjoint cases to consider. **a).** The volume of those $x \in P_\rho$ for which $x \in P$ is $V(P) = 1$. **b).** The volume of those $x \in P_\rho$ for which $x \neq p(P, x)$

and $p(P,x)$ is a relative interior point of a facet of P is ρ times the surface area of P (this was a homework question). There are four facets of area 1 and two of area $\sqrt{2}$, so these points x have volume $(4+2\sqrt{2})\rho$. **c).** We now consider those points $x \in P_\rho$ for which $x \neq p(P,x)$ and $p(P,x)$ is a relative interior point of an edge of P. There are four regions of such points x for which $p(P,x)$ is in an edge parallel to L_1. These four regions can be translated to form a set whose volume is that of a cylinder of radius ρ and length equal to the length of L_1. Their volume is therefore $\pi\rho^2$. Similarly, the volume of those points for which the nearest point is a relative interior point of an edge parallel to L_2, or L_3 is $\pi\rho^2$ or $\pi\rho^2\sqrt{2}$, respectively. **d).** Finally, we consider those points $x \in P_\rho$ for which $x \neq p(P,x)$ and $p(P,x)$ is a vertex of P. The volume of such points x is the volume of a sphere of radius ρ and is therfore $4\pi\rho^2/3$. Putting these results together shows that the volume $V(P_\rho)$ of P_ρ is

$$V(P_\rho) = 1 + (4+2\sqrt{2})\rho + (2+\sqrt{2})\rho^2 + \frac{4\pi}{3}\rho^3.$$