CONVEXITY 2 SPRING 2000
HOMEWORK 5 – ANSWERS

Find V, S, M for each of the following bodies:

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1. A circular cylinder with radius r and height h.

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$$\begin{aligned} V(K+\rho B) &= \pi(r+\rho)^2 h + 2\pi \int_0^{\rho} \left(r + \sqrt{\rho^2 - x^2}\right)^2 dx \\ &= \pi(r+\rho)^2 h + 2\pi\rho(r^2+\rho^2) - \frac{2\pi}{3}\rho^3 + \pi^2 r \rho^2. \end{aligned}$$

It follows that

$$V(K) = \pi r^2 h \qquad S(K) = 2\pi r(r+h) \qquad M(K) = \pi(h + \pi r).$$

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2. The body of revolution formed by rotating the shape considered in the class example 4 about the symmetry axis orthogonal to the one used in class.

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We consider the three parts (each taken twice) of $K + \rho B$, one of which is the frustrum F. We have

$$\begin{aligned} V(F) = \frac{\pi}{3}\Bigg[&\left(\frac{R+\rho}{\cos\alpha} - \rho\cos\alpha\right)\left(\frac{R}{\sin\alpha} + \rho\sin\alpha\right)^2 \\ &- \left(\frac{R+\rho}{\cos\alpha} - (R+\rho)\cos\alpha\right)(R+\rho)^2\sin^2\alpha\Bigg] \\ = \frac{\pi}{3}\Bigg[&\frac{1-\sin^6\alpha}{\cos\alpha\sin^2\alpha}R^3 + \frac{3(1-\sin^4\alpha)}{\cos\alpha}R^2\rho + 3R\rho^2\sin^2\alpha\cos\alpha\Bigg]. \end{aligned}$$

Then

$$\begin{aligned} V(K+\rho B) &= 2\pi \int_0^{\rho\cos\alpha} \left[\frac{R}{\sin\alpha} + \sqrt{\rho^2 - x^2}\right]^2 dx \\ &\qquad + 2\pi \int_{(R+\rho)\cos\alpha}^{R+\rho} \left[(R+\rho)^2 - x^2\right] dx + 2V(F) \\ &= 2\pi \left[\left(\frac{R^2}{\sin^2\alpha} + \rho^2\right)\rho\cos\alpha - \frac{1}{3}\rho^3\cos^3\alpha + \frac{4\pi R}{\sin\alpha}\int_0^{\rho\cos\alpha}\sqrt{\rho^2 - x^2}\,dx\right] \\ &\quad + 2\pi\left[(R+\rho)^3(1-\cos\alpha) - \frac{1}{3}(R+\rho)^3(1-\cos^3\alpha)\right] + 2V(F). \end{aligned}$$

Note that

$$\int_0^{\rho\cos\alpha}\sqrt{\rho^2 - x^2}\,dx = \frac{\rho^2}{2}\left[\frac{\pi}{2} - \alpha + \sin\alpha\cos\alpha\right].$$

Equating coefficients gives

$$\begin{aligned} V(K) &= \frac{2\pi R^3}{3}\left[3(1-\cos\alpha) - (1-\cos^3\alpha) + \frac{1-\sin^6\alpha}{\cos\alpha\sin^2\alpha}\right] \\ &= \frac{2\pi R^3}{3}\left[2 + \frac{\cos^3\alpha}{\sin^2\alpha}\right]; \\ S(K) &= 2\pi R^2\left[\frac{\cos\alpha}{\sin^2\alpha} + 3(1-\cos\alpha) - (1-\cos^3\alpha) + \frac{1-\sin^4\alpha}{\cos\alpha}\right] \\ &= 2\pi R^2\left[2 + \frac{\cos^3\alpha}{\sin^2\alpha}\right]; \\ M(K) &= 2\pi R\left[\frac{\pi/2 - \alpha + \sin\alpha\cos\alpha}{\sin\alpha} + 3(1-\cos\alpha) - (1-\cos^3\alpha) + \cos\alpha\sin^2\alpha\right] \\ &= 2\pi R\left[2 - \cos\alpha + \frac{\pi/2 - \alpha}{\sin\alpha}\right]. \end{aligned}$$

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3. The body of revolution formed by rotating a sector of a circle of radius R and angle 2α about its symmetry axis.

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We split $K + \rho B$ into four bodies, one of which is a frustrum F. We have

$$\begin{aligned} V(F) &= \frac{\pi}{3}\left[\left(R\cos\alpha - \rho\sin\alpha + \frac{\rho}{\sin\alpha}\right)(R\sin\alpha + \rho\cos\alpha)^2\right. \\ &\qquad\qquad \left. - \left(\frac{\rho}{\sin\alpha} - \rho\sin\alpha\right)\rho^2\cos^2\alpha\right] \\ &= \frac{\pi}{3}\left[R^3\sin^2\alpha\cos\alpha + 3R^2\rho\sin\alpha\cos^2\alpha + 3R\rho^2\cos^3\alpha\right]. \end{aligned}$$

We have

$$\begin{aligned}
V(K+\rho B) &= \pi \int_{-\rho}^{-\rho\sin\alpha} (\rho^2 - x^2)\,dx \\
&\qquad + \pi \int_{R\cos\alpha-\rho\sin\alpha}^{(R+\rho)\cos\alpha} \left[R\sin\alpha + \sqrt{\rho^2-(x-R\cos\alpha)^2}\right]^2 dx \\
&\qquad\qquad + \pi \int_{(R+\rho)\cos\alpha}^{R+\rho} \left[(R+\rho)^2 - x^2\right]\,dx + V(F) \\
&= \pi\left[\rho^3(1-\sin\alpha) - \frac{1}{3}\rho^3(1-\sin^3\alpha)\right] \\
&\qquad + 2\pi R\sin\alpha \int_{R\cos\alpha-\rho\sin\alpha}^{(R+\rho)\cos\alpha} \sqrt{\rho^2-(x-R\cos\alpha)^2}\,dx \\
&\qquad + \pi\Big[\big(R^2\sin^2\alpha + \rho^2 - R^2\cos^2\alpha\big)\,\rho(\cos\alpha+\sin\alpha) \\
&\qquad + R\cos\alpha\,\big((R+\rho)^2\cos^2\alpha - (R\cos\alpha-\rho\sin\alpha)^2\big) \\
&\qquad - \frac{1}{3}\,\big((R+\rho)^3\cos^3\alpha - (R\cos\alpha-\rho\sin\alpha)^3\big)\Big] \\
&\qquad + \pi\left[(R+\rho)^3(1-\cos\alpha) - \frac{1}{3}(R+\rho)^3(1-\cos^3\alpha)\right] + V(F).
\end{aligned}$$

We note that

$$\int_{R\cos\alpha-\rho\sin\alpha}^{(R+\rho)\cos\alpha} \sqrt{\rho^2-(x-R\cos\alpha)^2}\,dx = \frac{\rho^2}{2}\left[\frac{\pi}{2} + 2\sin\alpha\cos\alpha\right].$$

Equating coefficients gives

$$\begin{aligned}
V(K) &= \pi R^3\left[1-\cos\alpha - \frac{1}{3}(1-\cos^3\alpha) + 3\sin^2\alpha\cos\alpha\right] \\
&= \pi R^3\left[\frac{2}{3} + 2\cos\alpha - \frac{8}{3}\cos^3\alpha\right]; \\
S(K) &= \pi R^2\Big[(\cos\alpha+\sin\alpha)(\sin^2\alpha-\cos^2\alpha) + 2\cos^3\alpha + 2\cos^2\alpha\sin\alpha \\
&\quad - \cos^3\alpha - \cos^2\sin\alpha + 3(1-\cos\alpha) - (1-\cos^3\alpha) + 3\sin\alpha\cos^2\alpha\Big] \\
&= \pi R^2\left[2 - 2\cos\alpha + 3\sin\alpha - 4\sin^3\alpha\right]; \\
M(K) &= \pi R\Bigg[\left(\frac{\pi}{2} + 2\sin\alpha\cos\alpha\right)\sin\alpha + \cos^3\alpha - \cos\alpha\sin^2\alpha - \cos^3\alpha \\
&\qquad\qquad + \cos\alpha\sin^2\alpha + 3(1-\cos\alpha) - (1-\cos^3\alpha) + 3\cos^3\alpha\Bigg] \\
&= \pi R\left[2 + \frac{\pi}{2}\sin\alpha - \cos\alpha + 2\cos^3\alpha\right].
\end{aligned}$$

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4. The regular dodecahedron with circumradius R.

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For the dodecahedron $p = 5,\ q = 3$ and so $k^2 = (3-\sqrt{5})/8$. The edge length L is therefore given by $L^2 = 2R^2(3-\sqrt{5})/3$. The dihedral angle ψ is given by

$$\sin\frac{\psi}{2} = \frac{\cos(\pi/3)}{\sin(\pi/5)}.$$

It follows that $\cos\psi = -1/\sqrt{5}$. The external angle, β, is $\pi - \psi$ and so $\beta = \tan^{-1} 2$. We deduce that

$$M(K) = \frac{30}{2} R\sqrt{\frac{2}{3}}\sqrt{3-\sqrt{5}}\tan^{-1} 2 = 5\sqrt{3}\left(\sqrt{5}-1\right) R\tan^{-1} 2.$$

The area of one of the pentagonal faces is

$$\frac{5}{2}\frac{L^2}{4\sin^2(\pi/5)}\sin\frac{2\pi}{5}.$$

It follows that

$$S(K) = \frac{15}{2}\frac{2R^2\left(3-\sqrt{5}\right)}{3}\frac{2\left(1+\sqrt{5}\right)}{4}\frac{2\sqrt{2}}{\sqrt{5-\sqrt{5}}} = 2R^2\sqrt{10}\sqrt{5-\sqrt{5}}.$$

For each of the Platonic solids, we have $V(K) = R_2 S(K)/3$, where R_2 denotes the inradius. For the dodecahedron, we have

$$\left(\frac{R_2}{R}\right)^2 = \left(\frac{\cos(\pi/3)\cot(\pi/5)}{\sin(\pi/3)}\right)^2 = \frac{5+2\sqrt{5}}{15}.$$

It follows that

$$(V(K))^2 = \frac{40}{81}\left(9+3\sqrt{5}\right)R^6 \qquad \text{and so} \qquad V(K) = \frac{2\sqrt{15}}{9}\left(1+\sqrt{5}\right)R^3.$$

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5. The regular icosahedron with circumradius R.

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For the icosahedron $p = 3$, $q = 5$ and so $k^2 = (3 - \sqrt{5})/8$. The edge length L is therefore given by $L^2 = 2R^2(5 - \sqrt{5})/5$. The dihedral angle ψ is given by

$$\sin\frac{\psi}{2} = \frac{\cos(\pi/5)}{\sin(\pi/3)}.$$

It follows that $\sin\psi = 2/3$, and so the external angle, β, is given by $\beta = \sin^{-1}(2/3)$. We deduce that

$$M(K) = \frac{30}{2}R\sqrt{\frac{2}{5}}\sqrt{5 - \sqrt{5}}\sin^{-1}\frac{2}{3} = 3R\sqrt{10}\sqrt{5 - \sqrt{5}}\sin^{-1}\frac{2}{3}.$$

The area of one of the triangular faces is

$$\frac{1}{2}L^2\sin\frac{2\pi}{3} = \frac{1}{5}\left(5 - \sqrt{5}\right)\frac{\sqrt{3}}{2}R^2.$$

It follows that

$$S(K) = 2\sqrt{3}\left(5 - \sqrt{5}\right)R^2.$$

For the icosahedron, we have

$$\left(\frac{R_2}{R}\right)^2 = \left(\frac{\cos(\pi/5)\cot(\pi/3)}{\sin(\pi/5)}\right)^2 = \frac{5 + 2\sqrt{5}}{15}.$$

It follows that

$$(V(K))^2 = \frac{8}{9}\left(5 + \sqrt{5}\right)R^6 \qquad \text{and so} \qquad V(K) = \frac{2}{3}\sqrt{2}\sqrt{5 + \sqrt{5}}R^3.$$