CONVEXITY 2 SPRING 2000
HOMEWORK 4 – ANSWERS

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1. Let Q be an $(n-1)$-polytope in $\mathbb{E}^n$ with $o \in \text{relint}Q$ and let P be the bipyramid over Q defined by $P = \text{conv}(I \cup Q)$ where I is the line segment joining $\pm e_n$, the unit vectors orthogonal to aff$Q$. Describe the polar body P^*.

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We assume that aff$Q = \mathbb{E}^{n-1}$ and denote by Q^* the polar of Q in $\mathbb{E}^{n-1}$. We know from Theorem 21.7 that the vertices of P are precisley those of Q together with $\pm e_n$. It follows that

$$P^* = \{y \in \mathbb{E}^n : \langle y, e_n \rangle \leqslant 1\} \cap \{y \in \mathbb{E}^n : \langle y, e_n \rangle \geqslant -1\} \cap \bigcap_{\substack{v \\ \text{vertex of } Q}} \{y \in \mathbb{E}^n : \langle y, v \rangle \leqslant 1\}.$$

We also know that

$$Q^* = \bigcap_{\substack{v \\ \text{vertex of } Q}} \{y \in \mathbb{E}^{n-1} : \langle y, v \rangle \leqslant 1\}$$

It follows that $P^* = Q^* + I$ is a prism over Q^*.

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2. A 3-polytope is said to be simple if there are precisely three edges containing each vertex. Let P be a simple 3-polytope and let p_n denote the number of facets of P which are n-gons ($n = 3, 4, \dots$). Prove that

$$\sum_{n \geqslant 3} (6-n) p_n = 12.$$

Use similar techniques to show that all 3-polytopes (not just those that are simple) must have at least one n-gonal facet with $n \leqslant 5$.

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We denote by V, E, F the number of vertices, edges and facets of the 3-polytope P. For a simple polytope P, we have

$$\sum_{n \geqslant 3} p_n = F, \qquad \sum_{n \geqslant 3} n p_n = 2E = 3V.$$

Euler's formula yields $6V - 6E + 6F = 12$ which gives

$$\sum_{n \geqslant 3} (2n - 3n + 6) p_n = F \qquad \text{which is} \qquad \sum_{n \geqslant 3} (6-n) p_n = 12,$$

as required. For an arbitrary 3-polytope, we have

$$\sum_{n\geqslant 3} p_n = F, \qquad \sum_{n\geqslant 3} np_n = 2E, \qquad \sum_{n\geqslant 3} np_n \geqslant 3V,$$

since each vertex lies on at least three facets. The above reasoning then gives

$$\sum_{n\geqslant 3}(6-n)p_n \geqslant 12 \qquad \text{which is} \qquad 3p_3 + 2p_4 + p_5 \geqslant 12 + \sum_{n\geqslant 7}(n-6)p_n.$$

It is clear that the right side is positive and so at least one of $p_3,\ p_4,\ p_5$ must be positive, as required.

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3. Lay, Question 23.3

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We have $Q \subset K$ and so $K^* \subset Q^*$. For the reverse inclusion, let $x \in Q^*$ and $k \in K$. There are non-negative numbers $\lambda_1, \ldots, \lambda_{n+1}$ and points $q_1, \ldots, q_{n+1} \in Q$ such that

$$k = \lambda_1 q_1 + \cdots + \lambda_{n+1} q_{n+1} \qquad \text{and} \qquad \lambda_1 + \cdots + \lambda_{n+1} = 1.$$

It follows that

$$\langle x, k\rangle = \lambda_1\langle x, q_1\rangle + \cdots + \lambda_{n+1}\langle x, q_{n+1}\rangle \leqslant \lambda_1 + \cdots + \lambda_{n+1} = 1,$$

and so $x \in K^*$, as required.

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4. Lay, Question 23.5

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a). We know from Theorem 23.3(b) that K^* is closed convex and contains the origin. Consequently Theorem 23.5 gives $K^{***} = K^*$.

b). First, we note from (the proof of) Theorem 23.5 that $K \subset K^{**}$ for all sets K. Next, if K is bounded we can choose $R > 0$ so that $K \subset RB$ in which case $(1/R)B \subset K^*$ which implies $o \in \mathrm{int}K^*$. Conversely, if $o \in \mathrm{int}K^*$, we can find an $r > 0$ such that $rB \subset K^*$ in which case our previous observation gives $K \subset K^{**} \subset (1/r)B$, as required.

c). Let $x \in K^*$ and $y \in \mathrm{conv}K$, there are non-negative numbers $\lambda_1, \ldots, \lambda_{n+1}$ and points $k_1, \ldots, k_{n+1} \in K$ such that

$$y = \lambda_1 k_1 + \cdots + \lambda_{n+1} k_{n+1} \qquad \text{and} \qquad \lambda_1 + \cdots + \lambda_{n+1} = 1.$$

It follows that

$$\langle x, y\rangle = \lambda_1\langle x, k_1\rangle + \cdots + \lambda_{n+1}\langle x, k_{n+1}\rangle \leqslant \lambda_1 + \cdots + \lambda_{n+1} = 1,$$

and so $x \in (\text{conv}K)^*$. Thus $K^* \subset (\text{conv}K)^*$ for all K. Now assume that $o \in \text{int}\,\text{conv}K$. We can find $r > 0$ such that $rB \subset \text{conv}K$ and so $K^* \subset (\text{conv}K)^* \subset (1/r)B$, and therefore K^* is bounded.

For the reverse implication, assume $o \notin \text{int}\,\text{conv}K$. It follows from Theorem 4.5 that there is a hyperplane through o which does not meet $\text{int}\,\text{conv}K$. So we can choose $v \in \mathbb{E}^n$ such that $\langle v, k\rangle < 0$ for all $k \in \text{int}\,\text{conv}K$. It follows from Theorem 2.9 that $\langle v, k\rangle \leqslant 0$ for all $k \in \text{conv}K$ and so $\langle v, k\rangle \leqslant 0$ for all $k \in K$. Furthermore $\langle \alpha v, k\rangle \leqslant 0$ for all $\alpha > 0$ and all $k \in K$. Thus $\alpha v \in K^*$ for all $\alpha > 0$ and so K^* is unbounded.

e). The proof of Theorem 23.11 showed that $P^* = Q$ where Q is a polyhedral set. This part of the proof did not make use of the fact that P was n-dimensional nor that $o \in \text{int}P$, and so provides an answer to this question.

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5. Lay, Question 23.7

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First, let $x \in \bigcup_\alpha K_\alpha^*$. We have $\langle x, k\rangle \leqslant 1$ for all $k \in \bigcap_\alpha K_\alpha$ and so $\bigcup_\alpha K_\alpha^* \subset \left(\bigcap_\alpha K_\alpha\right)^*$. However, $\left(\bigcap_\alpha K_\alpha\right)^*$ is closed and convex, so $\text{cl}\,\text{conv}\bigcup_\alpha K_\alpha^* \subset \left(\bigcap_\alpha K_\alpha\right)^*$. For the reverse inclusion, assume $x \notin \text{cl}\,\text{conv}\bigcup_\alpha K_\alpha^*$. It follows from Theorem 4.9 that there is a $v \in \mathbb{E}^n$ such that $\langle x, v\rangle > 1$ and $\langle k, v\rangle < 1$ for all $k \in \text{cl}\,\text{conv}\bigcup_\alpha K_\alpha^*$ and therefore for all $k \in \bigcup_\alpha K_\alpha^*$. Consequently, $v \in \left(\bigcup_\alpha K_\alpha^*\right)^* = \bigcap_\alpha K_\alpha^{**} = \bigcap_\alpha K_\alpha$. It now follows that $x \notin \left(\bigcap_\alpha K_\alpha\right)^*$ and thus

$$\text{cl}\,\text{conv}\bigcup_\alpha K_\alpha^* = \left(\bigcap_\alpha K_\alpha\right)^*.$$