CONVEXITY 2 SPRING 2000
HOMEWORK 3 – ANSWERS

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1. Lay, Question 20.1

(10 points)

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a) A vertex is, by definition, an exposed point. The result then follows from the comments following Theorem 5.8

b) Let I be the segment $[0,1]$ and B be the unit ball, both in $\mathbb{E}^2$. Then the point $(1,1)$ is extreme but not a vertex of $I+B$.

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2. Lay, Question 20.2

(10 points)

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Note that, if H^+ is a closed half space in $\mathbb{E}^n$ then $L \cap H^+$ is either L or a closed half space in L. We assume P is a polytope and that $L \cap P \neq \emptyset$. Then P is a bounded polyhedral set and so there are closed half spaces $H_1^+, \ldots, H_k^+$ such that $P = \bigcap_{i=1}^k H_i^+$. Consequently $P \cap L = \bigcap_{i=1}^k (H_i^+ \cap L)$. If $\mathcal{S} = \{i : 1 \leqslant i \leqslant k,\ H_i^+ \cap L \neq L\}$. Then

$$P \cap L = \bigcap_{i \in \mathcal{S}} (H_i^+ \cap L),$$

is a bounded polyhedral set in L, and therefore a polytope.

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3. Lay, Question 20.8

(10 points)

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a) If we put $f_{124}(x,y,z) = \alpha x + \beta y + \gamma z$, we see that $3\alpha = 0,\ \alpha + \beta + 3\gamma = 0,\ 3\beta > 0$. It follows that $\alpha = 0,\ \beta = 3,\ \gamma = -1$ and so $f_{124}(x,y,z) = 3y - z$. Similar calculations give $f_{134}(x,y,z) = 3x - z$ and $f_{123}(x,y,z) = z$.

b) Note that $f_{14}(x,y,z) = 3x+3y-2z$. Also $f_{14}(a_1) = f_{14}(a_4) = 0,\ f_{14}(a_2) = f_{14}(a_3) > 0$ which gives the required result.

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4. Let X^4 be a four dimensional cross polytope.

a) How many vertices does X^4 have?
b) How many facets does X^4 have? Describe the nature of these facets.

(10 points)

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a) Let $x_1,\ldots,x_4$ be linearly independent vectors with $X^4 = \mathrm{conv}(\pm x_1,\ldots,\pm x_4)$. Let H be the subspace spanned by x_1, x_2, x_3. Then H is a hyperplane and we denote by v a normal vector with $\langle v, x_4\rangle > 0$. If $x \in X^4$ there are non-negative numbers $\lambda_1,\ldots,\lambda_4,\mu_1,\ldots,\mu_4$ such that

$$x = \sum_{i=1}^{4}(\lambda_i - \mu_i)x_i \qquad \text{and} \qquad \sum_{i=1}^{4}(\lambda_i + \mu_i) = 1.$$

It follows that

$$\langle x, v\rangle = \langle\sum_{i=1}^{4}(\lambda_i - \mu_i)x_i, v\rangle = (\lambda_4 - \mu_4)\langle x_4, v\rangle \leqslant \langle x_4, v\rangle,$$

with equality if and only if $x = x_4$ that is $\lambda_4 = 1$ and all other λ's and μ's are zero. We deduce that x_4 is a vertex, in fact all the points $\pm x_1,\ldots,\pm x_4$ are vertices. The definition of X^4 then shows that these are precisely all the vertices and so there are eight.

b) The facets of X^4 are convex hulls of the facets of X^3 with either of $\pm x_4$. Therefore there are twice as many facets of X^4 as there are of X^3. This argument could be repeated to show that the total number of facets is sixteen. In addition, if the facets of X^3 were 2-simplices then the facets of X^4 are 3-simplices. An analogous argument shows that the facets of X^3 are indeed 2-simplices and so we are finished.

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5. Let M be the moment curve in $\mathbb{E}^4$, that is $M = \{(t,t^2,t^3,t^4) \in \mathbb{E}^4 : t \in \mathbb{R}\}$. Let V be any finite subset of M and put $P = \mathrm{conv}V$. Show that, if W is a subset of V comprising exactly four(distinct) points, then W is affinely independent. Deduce that every facet of P is a tetrahedron. Prove that every two points of V are vertices of P which are joined by an edge of P. Polytopes with this property are said to be neighbourly. Not that, in dimension 3, the only neighbourly polytopes are tetrahedra.

(20 points)

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If there were four affinely dependnt points on M there would be numbers x, y, z, w not all zero and numbers a, b, c, d such that

$$\begin{aligned} x + y + z + w &= 0 \\ ax + by + cz + dw &= 0 \\ a^2x + b^2y + c^2z + d^2w &= 0 \\ a^3x + b^3y + c^3z + d^3w &= 0 \\ a^4x + b^4y + c^4z + d^4w &= 0 \end{aligned}$$

The first four of these equations have determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}.$$

This is a Vandermonde determinant and equals $(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$. We are assuming that the a, b, c, d are all distinct and so the only solution of the first four equations is $x = y = z = w = 0$. Thus the points are affinely indepependent. More or less the same argument shows that any five points on the moment curve are affinely independent (which is what I meant to ask you). If F is a facet of P then there is a support hyperplane H such that $P \cap H = F$. Note that any hyperplane can only meet the moment curve in at most four points since a polynomial of degree four has at most four real roots. Clearly our support plane H meets M in exactly four points. These are necessarily affinely independent and therefore form the vertices of a tetrahedron, as required. Let $\mathbf{v} = (v, v^2, v^3, v^4), \mathbf{w} = (w, w^2, w^3, w^4) \in V$ and assume that $v < w$. Now choose $\mathbf{p} = (p, p^2, p^3, p^4), \mathbf{q} = (q, q^2, q^3, q^4) \in M$ so that there are no points $(t, t^2, t^3, t^4) \in V$ with $v < t \leqslant p$ or with $q \leqslant t < w$. Let H be the hyperplane through $\mathbf{p}, \mathbf{q}, \mathbf{v}, \mathbf{w}$. The moment curve crosses H precisely at the points $\mathbf{p}, \mathbf{q}, \mathbf{v}, \mathbf{w}$. There are no points of V on M between $\mathbf{v}$ and $\mathbf{p}$ nor are there any between $\mathbf{q}$ and $\mathbf{w}$. Thus all points of V except $\mathbf{v}$, $\mathbf{w}$ lie strictly on one side of H. It follows that $\mathbf{v}$, $\mathbf{w}$ are the end points of an edge of P, as required.