CONVEXITY 2 SPRING 2000
HOMEWORK 2 – ANSWERS

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1. Assume x is a point not in the convex body K. Prove that the hyperplane containing $p(K, x)$ and normal to the vector $x - p(K, x)$ supports K.

(10 points)

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If this were not the case, there would be a point $k \in K$ such that $k = P(K, x) + v$ where $\langle v, x - p(K, x)\rangle > 0$. The convexity of K then shows that $y_\alpha = p(K, x) + \alpha v \in K$ for all $0 \leqslant \alpha \leqslant 1$. Then

$$\begin{aligned}\langle x - y_\alpha, x - y_\alpha\rangle &= \langle x - p(K, x) - \alpha v, x - p(K, x) - \alpha v\rangle \\ &= \langle x - p(K, x), x - p(K, x)\rangle - 2\alpha\langle v, x - p(K, x)\rangle + \alpha^2\langle v, v\rangle \\ &< \langle x - p(K, x), x - p(K, x)\rangle,\end{aligned}$$

for all sufficiently small $\alpha > 0$. This contradicts the fact that $P(K, x)$ is the closest point of K to x.

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2. Assume that r is any non-negative number and K is a convex body. Prove that $K + rB$ is the set of points x for which $d(x, p(K, x)) \leqslant r$.

(5 points)

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If $x \in K + rB$ then there is a $k \in K$, $0 \leqslant \lambda \leqslant r$ and a unit vector u such that $x = k + \lambda u$. Thus $d(x, p(K, x)) \leqslant d(x, k) = \lambda \leqslant r$. Conversely, if $d(x, p(K, x)) \leqslant r$ then $x \in rB + p(K, x) \subset K + rB$, as required.

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3. Let P be a polytope in $\mathbb{E}^n$ and let F be a facet of P. If $r \geqslant 0$, find an expression for the volume of the set of points $x \in P + rB$ for which $p(P, x) \in \text{relint} F$.

(10 points)

. . . .

If $x \in \text{relint} F$ and u is the outward unit normal to the support hyperplane corresponding to F then u is the only outward unit normal to P at x. It follows, from question 1, that

$$\{y \in \mathbb{E}^n : p(P, y) = x\} = \{x + \alpha u : \alpha \geqslant 0\}.$$

Furthermore, $d(x + \alpha u, x) = \alpha$ and so

$$\{y \in P + rB : p(P, y) \in \text{relint} F\} = \{x + \alpha u : 0 \leqslant \alpha \leqslant r,\ x \in \text{relint} F\}.$$

The latter is a right cylinder (or prism) of height r with base F. Its volume is therefore $rV_{n-1}(F)$ where V_{n-1} denotes $(n-1)$-dimensional volume.

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4. Lay, Question 19.3

(10 points)

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The ball of radius r and centre x in D^2 is the square with centre x and with sides parallel to the axes of length $2r$. It therefore suffices to find a non-convex set S such that, for all squares C of the above type with $C \cap S = bdC \cap S \neq \emptyset$, we have $S \cap C$ is a single point. A quarter circle of the form $\{(\cos\theta, \sin\theta) \in \mathbb{E}^2 : 0 \leqslant \theta \leqslant \pi/2\}$ is such an example.

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5. Lay, Question 19.4 c,d

(10 points)

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The ball of radius r and centre x in D^* is the square with centre x and with sides parallel to the lines $y = \pm x$ and of length $r\sqrt{2}$. If $e_1,\ e_2$ are the usual basis vectors then $x = e_1 + e_2$ is not in the unit ball B and each point of the segment e_1e_2 is distance 1 from x and this is the minimum distance from x to B. Part d is basically the same as question 19.3. This time an example is furnished by the quarter circle $\{(\cos\theta, \sin\theta) \in \mathbb{E}^2 : \pi/4 \leqslant \theta \leqslant 3\pi/4\}$.