CONVEXITY 2 SPRING 2000
HOMEWORK 1 – ANSWERS

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1. Let E be an ellipse with major and minor axes of lengths $2a$ and $2b$. Let F be the convex body obtained by rotating E through ninety degrees about its centre. Calculate the Hausdorff distance between E and F, giving full explanations

(10 points)

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The support functions of E and F are given by

$$h_E(u) = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} \qquad \text{and} \qquad h_F(u) = \sqrt{b^2\cos^2\theta + a^2\sin^2\theta},$$

where $u = (\cos\theta, \sin\theta)$. Clearly the maximum of h_E corresponds to $\theta = 0$ (assuming $a \geqslant b$). The minimum of h_F occurs at the same value of θ. Hence

$$D(E, F) = \max_{\|u\|=1} |h_E(u) - h_F(u)| = |a - b|.$$

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2. Let $H_1, H_2, \ldots$ be a sequence of supporting hyperplanes to a convex body K contained in the ball rB. Assume that the sequence of intersections $H_i \cap rB$ converges. Prove that, if $R > r$, the sequence of intersections $H_i \cap RB$ also converges. If the limit of this sequence is denoted by $T(R)$, prove that $T(R)$ is the intersection of a support hyperplane of K with RB.

(20 points)

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For $i = 1, 2, \ldots$, let u_i be the outer unit normal to H_i and let D_i denote the unit ball (of dimension $n-1$) in the subspace $u_i^{\perp}$. Then

$$H_i \cap rB = \sqrt{r^2 - h_K^2(u_i)}D_i + h_K(u_i)u_i.$$

There must be a convergent subsequence, $\{u_{i_j}\}_{j=1}^{\infty}$ say, of $\{u_i\}_{i=1}^{\infty}$. If the limit of this subsequence is u_0 then $D_{i_j} \to D_0$ as $j \to \infty$ because

$$D(D_i, D_0) < \sqrt{\frac{1}{\langle u_i, u_0\rangle^2} - 1}.$$

Hence $H_{i_j} \cap rB \to \sqrt{r^2 - h_K^2(u_0)}D_0 + h_K(u_0)u_0$ as $j \to \infty$. In fact this must be the limit of the $H_i \cap rB$ as $i \to \infty$. Thus

$$h_i \cap RB \to \sqrt{R^2 - h_K^2(u_0)}D_0 + h_K(u_0)u_0 = H_0 \cap RB \qquad \text{as } i \to \infty.$$

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3. Assume that the convex bodies $K_1, K_2, \ldots$ converge to K. Let $x_i \in K_i$ for each $i = 1, 2, \ldots$. Show that $\{x_i\}_{i=1}^{\infty}$ contains a subsequence convergent to a point of K.

(10 points)

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The sequence $\{x_i\}_{i=1}^{\infty}$ is bounded because the sequence $\{K_i\}_{i=1}^{\infty}$ converges and is therefore bounded. So we may choose a convergent subsequence $\{x_{i_j}\}_{j=1}^{\infty}$ with limit x say. Given n we may choose $N(n)$ so that

$$d(x_{i_j}, x) < \frac{1}{2n} \qquad \text{and} \qquad K_{i_j} \subset K + \frac{1}{2n}B \qquad \text{if } j \geqslant N(n).$$

Consequently, for each $n = 1, 2, \ldots$, we may choose $k_n \in K$ such that $d(k_n, x_{i_{N(n)}}) \leqslant 1/2n$. It follows that $d(k_n, x) < 1/n$ and so x is a limit point of K and therefore a point of K.

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4. Prove that a convergent sequence of balls must converge either to a ball or to a point.

(10 points)

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Assume the i-th ball, B_i, of the sequence is $r_iB + x_i$ then $D(B_i, B_j) = d(x_i, x_j) + |r_i - r_j|$. Consequently the sequences $\{x_i\}_{i=1}^{\infty}$ and $\{r_i\}_{i=1}^{\infty}$ are both convergent (because they must both be Cauchy). We let the limits be x and $r \geqslant 0$. The limit of the sequence of balls is therefore $rB + x$. If $r = 0$, this is a point, otherwise it is a ball.

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5. Assume that the convex bodies $K_1, K_2, \ldots$ converge to K. If each K_i contains the convex body Q and is contained in the convex body P, prove that $Q \subset K \subset P$.

(10 points)

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We have $h_Q \leqslant h_{K_i} \leqslant h_P$ for all i. Now $h_{K_i} \to h_K$ as $i \to \infty$. Hence $h_Q \leqslant h_K \leqslant h_P$ and so $Q \subset K \subset P$.

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6. Assume that the convex bodies $K_1, K_2, \ldots$ converge to K. Assume further that S is a closed set which does not intersect the interior of any K_i. Prove that S does not intersect the interior of K.

(10 points)

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Let $\delta > 0$ and choose I so that $K \subset K_i + (\delta/2)B$ if $i \geqslant I$. Next assume $B(x, \delta) \subset K$. Then

$$h_{B(x,\delta)}(u) = \langle x, u\rangle + \delta \leqslant h_K(u) \leqslant h_{K_i}(u) + \frac{\delta}{2} \qquad \text{for all } \|u\| = 1 \text{ and } i \geqslant I.$$

Thus $\langle x, u\rangle + \delta/2 \leqslant h_{K_i}(u)$ for all such u and $i \geqslant I$. That is

$$\langle x, u\rangle + \frac{\delta}{2} \leqslant h_{K_i}(u) \qquad \text{and so} \qquad h_{x+(\delta/2)B} \leqslant h_{K_i}.$$

Thus $B(x, \delta/2) \subset K_i$ for all $i \geqslant I$. It follows that each interior point of K is an interior point of K_i for sufficiently large i. This gives the required result.