Section 4. Separating Hyperplanes

If f is a linear functional on $\mathbb{E}^n$ and A is a subset of $\mathbb{E}^n$, we write $f(A) \leq \alpha$ if $f(x) \leq \alpha$ for all $x \in A$; similar notation is used for the other inequalities.

The hyperplane $[f : \alpha]$ *separates* the sets A and B if either

i) $f(A) \leq \alpha$ and $f(B) \geq \alpha$; or
ii) $f(A) \geq \alpha$ and $f(B) \leq \alpha$.

The hyperplane $[f : \alpha]$ *strictly separates* the sets A and B if either

i) $f(A) > \alpha$ and $f(B) < \alpha$; or
ii) $f(A) < \alpha$ and $f(B) > \alpha$.

If A and B are externally tangent closed circular discs in the plane then they are separated by their common tangent line but not strictly separated.

Lemma 4.4. *Let S be an open convex subset of $\mathbb{E}^2$. If $x \in \mathbb{E}^2 \setminus S$, then there is a line L containing x such that $L \cap S = \phi$.*

Proof. Choose a coordinate system with x at the origin. Put

$$T = \{u \in \mathbb{E}^2 : \|u\| = 1 \text{ and } x + \alpha u \in S, \text{ for some } \alpha > 0\}.$$

The openness of S shows that if $u \in T$ there is a $\delta > 0$ such that, if $\|v\| = 1$ and $\|u - v\| < \delta$, then $v \in T$. The convexity of S shows that if $u, v \in T$, then the shorter of the two arcs of the unit circle joining u and v is also contained in T. [Notice that if u and $-u$ are both in T then $x \in S$]. It follows that T is an open subarc of the unit circle. The total angle that this arc subtends at the origin is at most π; since $x \notin S$. The line L through x and an endpoint of this arc satisfies the theorem. □

Theorem 4.5. *Let F be a k-dimensional flat and S an open convex subset of $\mathbb{E}^n$ such that $F \cap S = \phi$. If $0 \leq k \leq n-2$, there is a flat F^* of dimension $k+1$ such that $F^* \supset F$ and $F^* \cap S = \phi$.*

Proof. Let V be the subspace of $\mathbb{E}^n$ orthogonal to F. Then $\dim V \geq 2$. Let $\pi : \mathbb{E}^n \to V$ denote the orthogonal projection onto V. Note that πF is a point of V and πS is an open convex subset of V with $\pi F \notin \pi S$. We aim to show that there is a line L in V with $\pi F \in L$ and $L \cap \pi S = \phi$. Once this is established then $F^* = \pi^{-1} L$ is a $(k+1)$-dimensional flat satisfying our requirements. We establish the existence of L by contradiction. Assume there is no such line L. Let L_1 and L_2 be orthogonal lines in V through πF, these must meet πS. Denote by V_1 the plane spanned by L_1 and L_2. Then $V_1 \cap \pi S$ is an open 2-dimensional convex subset of the plane V_1. It follows from Lemma 4.4 that there is a line L in V_1 such that$\pi F \in L$ and $L \cap (V_1 \cap \pi S) = \phi$. But $L \cap V_1 = L$ and so L is the line we are seeking. □

Corollary 4.6. *Let S be an open convex subset of $\mathbb{E}^n$ and let F be a k-dimensional flat $(0 \leq k < n)$ with $F \cap S = \phi$. Then there is a hyperplane H with $H \supset F$ and $H \cap S = \phi$.*

Theorem 4.7. *Suppose A and B are convex subsets of $\mathbb{E}^n$ with $\operatorname{int} A \neq \phi$. If $B \cap \operatorname{int} A = \phi$ there is a hyperplane that separates A and B.*

Proof. Note that $(\operatorname{int} A) - B$ is an open convex set in $\mathbb{E}^n$ which does not contain the origin. It follows from Corollary 4.6 that there is a hyperplane $H = [f : 0]$ with $H \cap [(\operatorname{int} A) - B] = \phi$;

f is a non-trivial linear functional on $\mathbb{E}^n$. We can assume, without loss of generality, that $f(\operatorname{int} A) - B) > 0$. Now let $x \in A$, $y \in B$, and $x_0 \in \operatorname{int} A$. We know from Theorem 2.9 that

$$x_n = \frac{1}{n}x_0 + \left(\frac{n-1}{n}\right)x \in \operatorname{relint}\overline{x_0x} \subset \operatorname{int} A.$$

Then $0 < f(x_0 - y) = f(x_n) - f(y)$. Furthermore,

$$f(x) = \lim_{n \longrightarrow \infty} f(x_n) \geq f(y).$$

Consequently, if $\alpha = \inf\{f(x) : x \in A\}$, the sets A and B are separated by $[f : \alpha]$. □

Note that if A and B are closed line segments in $\mathbb{E}^2$ with relative interiors which intersect at the origin and lie in different lines, then $\operatorname{int} A = \operatorname{int} B = \phi$. However, there is no line that separates them.

If $H = [f : \alpha]$ the corresponding closed and open *half-spaces* are the sets

$$\begin{array}{lll} \{x \in \mathbb{E}^n : f(x) \geq \alpha\} & \text{and} & \{x \in \mathbb{E}^n : f(x) \leq \alpha\} \quad \text{(closed)} \\ \{x \in \mathbb{E}^n : f(x) > \alpha\} & \text{and} & \{x \in \mathbb{E}^n : f(x) < \alpha\} \quad \text{(open)} \end{array}$$

If S is a subset of $\mathbb{E}^n$ and H is a hyperplane, we say that H *bounds* S if S is contained in one of the closed half-spaces determined by H; otherwise H is said to *cut* S. Note that H cuts S if and only if S is a non-empty intersection with both open half-spaces determined by H.

Lemma 4.10. *A hyperplane H cuts the convex set S if and only if S is not a subset of H and $H \cap \operatorname{relint} S \neq \phi$.*

Proof. Let $H = [f : \alpha]$ and $z \in \operatorname{relint} S$. Assume H cuts S, clearly S is not a subset of H. We can choose $x, y \in S$ with $f(x) > \alpha$ and $f(y) < \alpha$. If $z \in H$, we are finished. Otherwise assume, without loss of generality, $f(z) > \alpha$. Note that there is a $p \in \operatorname{relint}\overline{yz}$ with $f(p) = \alpha$; in fact

$$p = \left(\frac{f(z) - \alpha}{f(z) - f(y)}\right) y + \left(\frac{\alpha - f(y)}{f(z) - f(y)}\right) z.$$

Conversely, assume S is not a subset of H and $H \cap \operatorname{relint} S \neq \phi$. Assume $\dim S = k$ and put $J = \operatorname{aff} S$. Since $J \not\subset H$, $J \cap H$ is a hyperplane in J. Now there is a ball $B(p, \delta) \cap J \subset S$, since $J \cap H$ contains the midpoint of this ball, there are points of the ball on both sides of $J \cap H$. Consequently, S meets both open half-spaces determined by H. □

Theorem 4.11. *Suppose A and B are convex subsets of $\mathbb{E}^n$ such that $\dim(A \cup B) = n$. Then A and B can be separated by a hyperplane if and only if $\operatorname{relint} A \cap \operatorname{relint} B = \phi$.*

Proof. If $x \in \operatorname{relint} A \cap \operatorname{relint} B$, then any separating hyperplane H would have to contain x. Furthermore, if $A \subset H$ and $B \subset H$ then $\dim(A \cup B) < n$. It follows therefore, from Lemma 4.10 that no such separating hyperplane exists.

For the converse, we assume that $\operatorname{relint} A \cap \operatorname{relint} B = \phi$ and construct a separating hyperplane. If $\dim A = n$, then $\operatorname{int} A \neq \phi$. Note that $\operatorname{relint} B$ is convex and so Theorem 4.7 shows that there is a hyperplane H separating A and $\operatorname{relint} B$. Consequently, A and $\operatorname{relint} B$ lie in opposite closed half-spaces determined by H. It follows from Theorem 2.9 that A and B lie in opposite closed half-spaces determined by H. The result is therefore established if either one of the bodies has dimension n. We now assume inductively that the result is established in the case that either body has dimension at least k, for some $k \leq n$. We further assume that $\dim A = k-1$. We can choose a hyperplane J containing A, and a line segment I such that $I \not\subset J$ (and not parallel to J). If x and y are the endpoints of I then $A + \frac{x+y}{2}$ is a $(k-1)$-dimensional convex body lying in the hyperplane $J + \frac{x+y}{2}$. Also we can write

$$A + I = C \cup D$$

where C and D are convex sets lying in opposite closed half-spaces determined by $J + \frac{x+y}{2}$. Furthermore $A + I$ is of dimension k, as are C and D. We note that we cannot have

$$\operatorname{relint} C \cap \left(\operatorname{relint} B + \frac{x+y}{2}\right) \neq \phi \qquad \text{and} \qquad \operatorname{relint} D \cap \left(\operatorname{relint} B + \frac{x+y}{2}\right) \neq \phi.$$

Without loss of generality, we may assume

$$A + x \subset C \qquad \text{and} \qquad A + y \subset D.$$

We have, if the above sets were both non-empty,

$$a_1 + \mu x + (1-\mu)y = b_1 + \frac{x+y}{2}$$

where $a_1 \in \operatorname{relint} A$, $b_1 \in \operatorname{relint} B$, and $1/2 < \mu < 1$; and

$$a_2 + \lambda x + (1-\lambda)y = b_2 + \frac{x+y}{2}$$

where $a_2 \in \operatorname{relint} A$, $b_2 \in \operatorname{relint} B$, and $0 < \lambda < 1/2$. Since $\lambda < 1/2 < \mu$, we can choose $0 < \theta < 1$ such that

$$\theta\lambda + (1-\theta)\mu = \frac{1}{2}.$$

Then

$$\begin{aligned}
&\theta(b_2 + \frac{x+y}{2}) + (1-\theta)(b_1 + \frac{x+y}{2}) = \theta b_2 + (1-\theta)b_1 + \frac{x+y}{2} \\
&= (1-\theta)(a_1 + \mu x + (1-\mu)y) + \theta(a_2 + \lambda x + (1-\lambda)y) \\
&= (1-\theta)a_1 + \theta a_2 + \frac{x+y}{2}.
\end{aligned}$$

Consequently

$$(1-\theta)a_1 + \theta a_2 = (1-\theta)b_1 + \theta b_2$$

and so

$$\text{relint}\, A \cap \text{relint}\, B \neq \phi,$$

a contradiction. So we may assume, without loss of generality, that

$$\text{relint}\, C \cap (\text{relint}\, B + \frac{x+y}{2}) = \phi.$$

Now $\dim C = k$ and so by inductive assumption, there is a hyperplane H separating C and $B + \frac{x+y}{2}$. But $A + \frac{x+y}{2} \subset C$ and so $H - \frac{x+y}{2}$ separates A and B. □

Theorem 4.12. *Suppose A and B are non-empty convex sets with A compact and B closed. Then there is a hyperplane which strictly separates A and B if and only if A and B are disjoint.*

Proof. If there is a hyperplane strictly separating A and B then it follows immediately that A and B are disjoint.

Conversely, if A and B are disjoint then we know from the first homework assignment that $d(A, B) > 0$ where

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

Put $C = B(o, \frac{1}{2}d(A, B))$ then $A + C$ and $B + C$ are disjoint, open convex sets. It follows from Theorem 4.11 that they can be separated by a hyperplane H. It is now clear that A and B are strictly separated by H. □

Notice that compactness is essential in the above result. For example, if

$$A = \{(x, y) \in \mathbb{E}^2 : x \geq 0, y \geq \frac{1}{x}\} \qquad \text{and} \qquad B = \{(x, y) \in \mathbb{E}^2 : y = 0\}$$

then A and B are disjoint, closed convex sets which cannot be strictly separated.

Theorem 4.13. *Suppose A and B are non-empty compact sets. Then there is a hyperplane H which strictly separates A and B if and only if* $\text{conv}\, A \cap \text{conv}\, B = \phi$.

Proof. We know from Theorem 2.30 that $\text{conv}\, A$ and $\text{conv}\, B$ are compact. If they are disjoint, we may use Theorem 4.12 to find a hyperplane which strictly separates them. This same hyperplane strictly separates A and B.

Conversely, we suppose the hyperplane $H = [f : \alpha]$ strictly separates A and B with $f(A) > \alpha$ and $f(B) < \alpha$. If $x \in \text{conv}\, A$ there are points $a_1, \ldots, a_{n+1} \in A$ and $\lambda_1, \ldots, \lambda_{n+1} \geq 0$ such that

$$x = \lambda_1 a_1 + \cdots + \lambda_{n+1} a_{n+1} \qquad \text{and} \qquad \lambda_1 + \cdots + \lambda_{n+1} = 1.$$

Hence

$$f(x) = \lambda f(a_1) + \cdots + \lambda_{n+1} f(a_{n+1}) > (\lambda_1 + \cdots + \lambda_{n+1})\alpha = \alpha.$$

So $f(\text{conv}\, A) > \alpha$, similarly $f(\text{conv}\, B) < \alpha$. Consequently $\text{conv}\, A \cap \text{conv}\, B = \phi$. □

Theorem 4.14. *Suppose A and B are non-empty compact subsets of $\mathbb{E}^n$. There is a hyperplane strictly separating A and B if and only if for each set T of $n+1$ or fewer points of B, there is a hyperplane strictly separating A and T.*

Proof. In one direction this result is trivial. Conversely, assume that for each set T (above) there is a hyperplane strictly separating A and T. Now suppose that $y \in \operatorname{conv} B$. We can find $b_1, \ldots, b_{n+1} \in B$ and $\lambda_1, \ldots, \lambda_{n+1} \geq 0$ such that

$$y = \lambda_1 b_1 + \cdots + \lambda_{n+1} b_{n+1} \qquad \text{and} \qquad \lambda_1 + \cdots + \lambda_{n+1} = 1.$$

We let $H = [f : \alpha]$ be a hyperplane strictly separating A and $T = \{b_1, \ldots, b_{n+1}\}$ with $f(A) > \alpha$ and $f(T) < \alpha$. As in the proof of Theorem 4.13 we have $f(y) < \alpha$ and $f(\operatorname{conv} A) > \alpha$. Consequently, $y \notin \operatorname{conv} A$ and so $\operatorname{conv} A \cap \operatorname{conv} B = \phi$. Theorem 4.13 now provides the required strict separation. $\square$