Section 3. Hyperplanes and Linear Functionals

A hyperplane in $\mathbb{E}^n$ was already defined to be an $(n-1)$-dimensional flat; that is, a translate of an $(n-1)$-dimensional subspace of $\mathbb{E}^n$. A function $f : \mathbb{E}^n \longrightarrow \mathbb{E}^m$ is said to be *linear* if

$$f(x+y) = f(x) + f(y) \qquad \text{(additive)}$$
$$f(\lambda x) = \lambda f(x) \qquad \text{(homogeneous)}$$

for all $x, y \in \mathbb{E}^n$ and all $\lambda \in \mathbb{R}$.

A linear function $f : \mathbb{E}^n \longrightarrow \mathbb{R}$ is called a *linear functional*, in which case we denote by $[f : \alpha]$ the set

$$\{x \in \mathbb{E}^n : f(x) = \alpha\} \qquad (\alpha \in \mathbb{R}).$$

Theorem 3.2. *Suppose H is a subset of $\mathbb{E}^n$. Then H is a hyperplane if and only if there is a non-trivial linear functional f and a number δ such that $H = [f : \delta]$.*

Proof. First assume H is a hyperplane and let $x_0 \in H$. Then $V = H - x_0$ is the $(n-1)$-dimensional subspace of $\mathbb{E}^n$ parallel to H. We denote by v_0 a unit vector orthogonal to V. For each $x \in \mathbb{E}^n$, denote by x_V the orthogonal projection of x onto V (i.e. the nearest point of V to x). We know, from linear algebra, that there is a number α such that

$$x = x_V + \alpha v_0.$$

It is clear that α is uniquely determined by x. We define $f : \mathbb{E}^n \longrightarrow \mathbb{R}$ by $f(x) = \alpha$; so $|f|$ measures the distance of x to V.

Next we check that f is a linear functional. If $x, y \in \mathbb{E}^n$, we have

$$x = x_V + \alpha v_0 \qquad \text{and} \qquad y = y_V + \beta v_0$$

and so

$$x + y = x_V + y_V + (\alpha + \beta)v_0.$$

Consequently,

$$f(x+y) = \alpha + \beta = f(x) + f(y).$$

Also, if $\lambda \in \mathbb{R}$, $\lambda x = \lambda x_V + \lambda \alpha v_0$ and so $f(\lambda x) = \lambda \alpha = \lambda f(x)$. So f is a linear functional.

Finally we show that $H = [f : \delta]$ where $\delta = f(x_0)$. If $h \in H$ then $h = h_0 + v$ where $v \in V$. Consequently $f(h) = f(x_0) + f(v) = f(x_0)$. Now $x_0 = w_0 + \delta v_0$ where $w_0 \in V$ and so if $f(x) = \delta$ then $x = x_V + \delta v_0$ and therefore $x - x_0 = x_V - w_0 \in V$ and so $x \in H$ ($[f : \delta] \subset H$). So we have proved a non-trivial linear functional f and a number δ such that $H = [f : \delta]$.

For the converse, note that, since f is non-trivial, $f : \mathbb{E}^n \longrightarrow \mathbb{R}$ is surjective. Consequently, $\dim \ker f = n - 1$. Put $V = \ker f$, an $(n-1)$-dimensional subspace of $\mathbb{E}^n$. Now assume$f(x_0) = \delta$ and complete the proof by showing that $[f : \delta] = V + x_0$. If $v \in V$ then $f(v + x_0) = f(v) + f(x_0) = \delta$ and so $V + x_0 \subset [f : \delta]$. But $[f : \delta]$ is an affine set which is not the whole space. Consequently $\dim[f : \delta] \leq n - 1$ and so $[f : \delta] = V + x_0$. $\square$

Theorem 3.3. *If f and g are linear functionals on $\mathbb{E}^n$ such that $[f : \alpha] = [g : \beta]$ for some $\alpha, \beta \in \mathbb{R}$ then there is a number $\lambda \neq 0$ such that $f = \lambda g$ and $\alpha = \lambda\beta$.*

Proof. First assume g is trivial. Then $[g : \beta] = \mathbb{E}^n$ if $\beta = 0$ and $[g : \beta] = \phi$ if $\beta \neq 0$. So, if $\beta = 0$ we have $[f : \alpha] = \mathbb{E}^n$. Consequently $\alpha = f(0) = 0$ and f is trivial. In this case any $\lambda \neq 0$ works. If $\beta \neq 0$ we have $[f : \alpha] = \phi$ and so $\alpha \neq f(0) = 0$ and f is trivial since, if there were an $x \in \mathbb{E}^n$ with $f(x) \neq 0$ then $f(\frac{\alpha x}{f(x)}) = \alpha$, which is impossible. In this case put $\lambda = \alpha/\beta$.

Now assume g is not trivial and choose $x_0 \in \mathbb{E}^n$ with $g(x_0) \neq 0$. Put $\lambda = f(x_0)/g(x_0)$ and $V = [g : 0]$, the kernel of g. Note that V is an $(n-1)$-dimensional subspace $\mathbb{E}^n$. We have

$$v + \frac{\beta}{g(x_0)} x_0 \in [g : \beta] \qquad \text{for all } v \in V.$$

Hence

$$v + \frac{\beta}{g(x_0)} x_0 \in [f : \alpha] \qquad \text{for all } v \in V;$$

equivalently

$$f(v) + \frac{\beta}{g(x_0)} f(x_0) = \alpha \qquad \text{for all } v \in V.$$

Thus

$$f(v) + \lambda\beta = \alpha \qquad \text{for all } v \in V.$$

The fact that v is a subspace means $o \in V$ and therefore $\alpha = \lambda\beta$. Furthermore, if $x \in \mathbb{E}^n$, there is a $\mu \in \mathbb{R}$ such that $x = v + \mu x_0$ for some $v \in V$; this follows from the fact that $\dim V = n - 1$ and $x_0 \in V$. Hence

$$f(x) = f(r) + \mu f(x_0) = \mu f(x_0) = \mu\lambda g(x_0) = \lambda(g(v) + \mu g(x_0)) = \lambda g(x)$$

as required.

□

Theorem 3.4 and 3.5. *Let f be a linear functional defined on $\mathbb{E}^n$.*

a) *There is a $z \in \mathbb{E}^n$ such that $f(x) = \langle x, z\rangle$ for all $x \in \mathbb{E}^n$;*
b) *f is continuous;*
c) *Each set of $[f : \alpha]$ is closed and therefore every hyperplane is closed.*

Proof.

a) If f is trivial put $z = o$. Otherwise put $V = [f : 0]$. Then V is an $(n-1)$-dimensional subspace of $\mathbb{E}^n$ and we may choose a vector u orthogonal to V. We put $g(x) = \langle x, u\rangle$ for each $x \in \mathbb{E}^n$. Then g is a linear functional on $\mathbb{E}^n$ and $[f : 0] = [g : 0]$. It follows from Theorem 3.3 that there is a $\lambda \in \mathbb{R}$ with $f = \lambda g$. If we put $z = \lambda u$ then $f(x) = \langle x, z\rangle$ for each $x \in \mathbb{E}^n$.
b) It follows from a) that

$$|f(x) - f(y)| = |\langle x - y, z\rangle| \leq \|x - y\| \, \|z\| \text{ for all } x, y \in \mathbb{E}^n.$$

So if $\varepsilon > 0$ is given, we choose $\delta > 0$ so that

$$\delta \|z\| < \varepsilon.$$

Then, if $y \in B(x, \delta)$ we have $f(y) \in B(f(x), \varepsilon)$. Consequently f is continuous.

c) It follows from b) that

$$\{x : f(x) > \alpha\} \cup \{x : f(x) < \alpha\} = f^{-1}(\alpha, \infty) \cup f^{-1}(-\infty, \alpha,)$$

is open. Now $[f : \alpha]$ is the complement of this set and must therefore be closed. We learned in Theorem 3.2 that each hyperplane is of the form $[f : \alpha]$ for some linear functional f and some member α. So each hyperplane is closed.

□

We note that we have shown that if H is a hyperplane then there is a $\gamma \in \mathbb{R}$ and a vector $z \in \mathbb{E}^n$ such that

$$H = \{x \in \mathbb{E}^n : \langle x, z \rangle = \gamma\}.$$

We also know that z is orthogonal to all vectors parallel to H since it is orthogonal to all vectors in the subspace parallel to H. We could choose z to be a unit vector, in which case γ measures that distance from o to H.