

Problems 1, 3, 4, 8 from Section 2.1 of the book.

Hint for Problem 2.1/3: How is the existence of $\lim_{n \rightarrow \infty} f_n(x)$ related to the numbers $\limsup f_n(x)$ and $\liminf f_n(x)$? Having answered this question, use Proposition 2.7.

Additional problem 1. Consider the sequence of functions

$$f_j : [0, 1] \rightarrow \mathbb{R} : x \mapsto f(x) = x^j, \quad j \in \mathbb{N}. \quad (1)$$

- (a) Using the ϵ - δ definition of continuity, prove that, for any $j \in \mathbb{N}$, the function f_j (given by equation (1)) is continuous.
- (b) Using the ϵ - N definition of convergence of sequences of real numbers, prove that the sequence $\{f_j(x)\}_{j=1}^{\infty}$ (given by equation (1)) is convergent for any $x \in [0, 1]$. Define the function $f : [0, 1] \rightarrow \mathbb{R}$ as $f(x) := \lim_{j \rightarrow \infty} f_j(x)$. What is f ? Is it continuous?
- (c) Directly from the definition of uniform convergence, demonstrate that the sequence $\{f_j(x)\}_{j=1}^{\infty}$ (given by equation (1)) does *not* converge uniformly on the interval $[0, 1]$.
- (c) Prove that if a sequence $\{f_j(x)\}_{j=1}^{\infty}$ of real-valued continuous functions defined on a set D converges uniformly on D to a function $f : D \rightarrow \mathbb{R}$, then f is continuous on D .

Additional problem 2. Let (X, \mathcal{M}) be a measurable space, and the functions $f_j : X \rightarrow \overline{\mathbb{R}}$ be measurable for all $j \in \mathbb{N}$. Give a direct proof of the fact that the function $\inf_j f_j$ is measurable.

Additional problem 3. Let (X, \mathcal{M}) be a measurable space, and $f_j : X \rightarrow \overline{\mathbb{R}}$ be Borel measurable functions for all $j \in \mathbb{N}$. Let $\lim_{j \rightarrow \infty} f_j(x)$ exist for all $x \in X$, and define the function $f : X \rightarrow \overline{\mathbb{R}}$ by $f(x) := \lim_{j \rightarrow \infty} f_j(x)$. Prove that the function f is Borel measurable.

Hint: Here is an idea: show that

$$f^{-1}((a, \infty]) = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} \left\{ x \in X : f_j(x) > a + \frac{1}{n} \right\},$$

and explain why this implies the Borel measurability of f .