MATH 4073 Homework 4 Due Fri, 09/24/10

Problem 1. Let the sequence $\{p_n\}_{n=1}^{\infty}$ be defined by $p_0 = 0$, $p_n = 1 - \frac{1}{2}\sin p_{n-1}$ for $n \geq 1$.

(a) Think of the sequence $\{p_n\}_{n=0}^{\infty}$ as a functional iteration, $p_n = g(p_{n-1})$, for an appropriate function g. Write g explicitly. Use Theorem 2.2 (page 54) to prove that g has a fixed point in the interval $[0, \frac{\pi}{2}]$.

(b) Looking at the explicit expression for $g'(x)$ and thinking about the slope of the graph of $g(x)$ (and comparing it with the slope of the "diagonal", $y = x$), give a simple argument showing that there cannot be more than one positive fixed point of g.

(c) Discuss the uniqueness of the fixed point using the Fixed Point Theorem (Theorem 2.3).

(d) In Mathematica, type the following code:

```
g[x_]=1 - Sin[x]/2;
p = 0;
For[ i = 1, i <= 130, i++,
  { p = g[p],
    Print[ i, "   ", N[p, 50]],
  }
]
exact = N[p,50]
```

then hold down SHIFT and press RETURN to execute it. The value of p at the last step is saved under the name `exact`; we will use it later to study the rate at which p_n approaches the fixed point of g in $[0, 1]$. What do you observe about the behavior of the sequence $\{p_n\}_{n=0}^{\infty}$?

Remark: To check that you indeed obtained the correct answer, you may type the commands `N[1-Sin[exact]/2,50]` and `N[exact,50]` (don't forget to hold down SHIFT and press RETURN to execute each command), and compare the two values. The command `N[`x`,`k`]` prints k digits of the number x.

(e) Now type in Mathematica (in the same session of Mathematica, so that it knows what the function g is)

```
p = 0;
For[ i = 0, i <= 130, i++,
  { pold = p, p = g[p],
    Print[i,"   ",N[p-exact,50],"   ",N[(p-exact)/(pold-exact),50]],
    }
  ]
```

At each step of the iteration, you see the value of the difference $(p_n - p)$ and of the ratio $\frac{p_n - p}{p_{n-1} - p}$ (p stands for the exact value of the fixed point). What do you observe about the behavior of these two sequences of numbers?

(f) The ratios $\frac{p_n - p}{p_{n-1} - p}$ in part (e) seem to converge to some number. What is the value of this number? (I want to see at least 16 decimal digits of it.) Could you predict this value without computing it? *Hint:* Look at the theorems discussed recently in class.

Problem 2. The real number $\sqrt[3]{a}$ can be thought of as the real root of the equation

$$x^3 = a \ . \qquad (1)$$

(a) For an appropriately chosen function f, rewrite (1) in the form $f(x) = 0$.

(b) Construct a Newton's iterative procedure for solving $f(x) = 0$ (from part (a)) numerically.

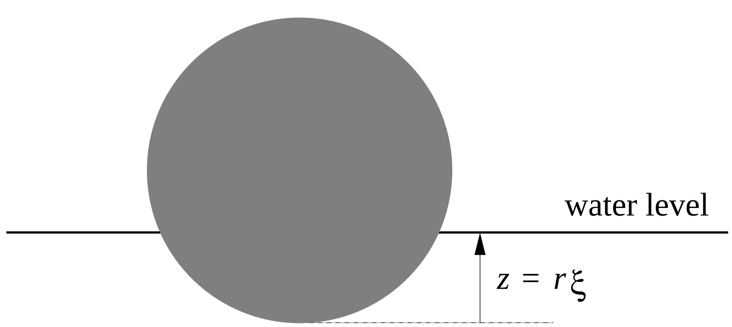
(c) Use Mathematica to implement the procedure developed in (b), and compute $\sqrt[3]{12}$ with accuracy of at least 10^{-100}. At each step of the iteration make Mathematica print the number of the step, the current value of p_n with 50 digits of accuracy, and the true error, $|p_n - \sqrt[3]{12}|$ with 10 digits of accuracy (of course, Mathematica knows how to compute cubic roots – to get $\sqrt[3]{12}$, type `12^(1/3)`). Look at the behavior of the error with n (the number of the step) – does the method look linearly or quadratically convergent? Explain how you came to this conclusion. Please attach your Mathematica printout.

(d) Did you expect the behavior observed in part (c)? Explain briefly what the general theory says (give the exact reference to the theorems you cite).

Problem 3. Directly from the definition, find the rates of convergence α and the asymptotic error constants λ for each of the sequences (all of which tend to 0)

$$(a)\ \ p_n = \frac{7}{n^3}\ ; \qquad (b)\ \ p_n = 3^{-n}\ ; \qquad (c)\ \ p_n = 10^{-5^n}\ .$$

Problem 4. A solid ball of radius r is floating in water, as shown in Figure 1.



The density of the ball is equal to ρ_{ball}, and the density of water is ρ_{water}. The ball is lighter than water, so it is floating in it; let z be the vertical distance from the lowest point of the ball to the surface of the water, as shown in the figure.

Using elementary integration, one can show that the part of the sphere submerged in the water has volume

$$V_{\text{submerged}} = \frac{\pi}{3} z^2 (3r - z) \tag{2}$$

(you do *not* need to prove this formula, I am sure you can to it).

According to law of Archimedes, the buoyant force pushing the ball upwards is equal to the weight of the water displaced by the body, i.e., the weight of the water which would have filled the submerged part of the body. If $g = 9.8 \, \frac{\text{m}}{\text{s}^2}$ stands for the free-fall acceleration, then the buoyant force is equal to

$$F_{\text{buoyant}} = g \, \rho_{\text{water}} \, V_{\text{submerged}} \ .$$

The ball will be in equilibrium if the weight of the ball,

$$F_{\text{weight of ball}} = g \, \rho_{\text{ball}} \, V_{\text{ball}} \ ,$$

(pulling the ball downwards) is equal to the buoyant force F_{buoyant} (which pushes the ball upwards):

$$g \, \rho_{\text{water}} \, V_{\text{submerged}} = g \, \rho_{\text{ball}} \, V_{\text{ball}} \ .$$

Canceling out g and using the formula for the volume of a ball, we obtain the following condition for equilibrium:

$$\rho_{\text{water}} \, \frac{\pi}{3} \, z^3 (3r - z) = \rho_{\text{ball}} \, \frac{4\pi}{3} r^3 \ .$$

Let s be a (positive) dimensionless quantity defined as

$$s = \frac{\rho_{\text{ball}}}{\rho_{\text{water}}} > 0 \ .$$

Using this definition, after some elementary algebra one can rewrite the equilibrium condition in the form

$$z^3 - 3rz^2 + 4sr^3 = 0 \ . \tag{3}$$

Finally, let us introduce the non-dimensional quantity

$$\xi = \frac{z}{r} \ ,$$

whose physically meaningful range is, obviously, $\xi \in [0, 2]$. In terms of ξ, the equilibrium condition (3) takes the simple form

$$\xi^3 - 3\xi^2 + 4s = 0 \ . \tag{4}$$

(a) Check that the expression (2) for the volume of the submerged part is plausible. In other words, do *not* prove it, but think of several – I want you to think of three – different cases in which this formula should give obvious results (for example, it is obvious what $V_{\text{submerged}}$ should be if $z = 0$ – check that (2) indeed gives you this value for $V_{\text{submerged}}$; then think of two more obvious cases).

(b) It is clear from the physics of the problem that for s in the physically meaningful range, the equation (4) must have a unique solution ξ^* in the physically meaningful range of values of ξ. You have to show mathematically that this is indeed the case.

To do this, I suggest that you do the following. Let $f(\xi) = \xi^3 - 3\xi^2 + 4s$ stand for the left-hand side of the equilibrium condition (4). The function f is a cubic polynomial, hence it cannot have more than three real zeros, and can have no more than two extrema. Show that the function f always has a local maximum at the point $\xi_1 = 0$. What is the value ξ_2 of ξ for which f has a local minimum? What are the values of $f(\xi_1)$ and $f(\xi_2)$ if s is such that the ball floats (i.e., does not sink)? Is the function f continuous? Use all these facts to show mathematically (by using some of the theorems we have discusses in class) that if s in the "floating" range, the equation $f(\xi) = 0$ has a unique physically meaningful solution. Which theorem have you used to draw this conclusion?

(c) Use the MATLAB program `newton.m` to apply Newton's method for computing the numerical value of ξ^* for $s = 0.2$, 0.4, 0.5 (this is obvious!), 0.6, and 0.8, with tolerance 10^{-8}; start from some reasonable initial value. Write your results in a table containing the value of x, the corresponding values of ξ^*, and the number of steps that Newton's method needed to provide the necessary precision. If the value of ξ^* you obtain is not physically reasonable, change the initial point (remember, Newton's method gets lost very easily).

Please attach the printout from running the program.

(d) Continue your reasoning in part (b) to show that if $s > 1$, there is only one real solution of the equilibrium condition (4), and this solution is not physically reasonable.