Universal scalings of universal scaling exponents

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Abstract. In the last decades, renormalization group (RG) ideas have been applied to describe universal properties of different routes to chaos (quasi-periodic, period doubling or tripling, Siegel disk boundaries, etc.). Each of the RG theories leads to universal scaling exponents which are related to the action of certain RG operators.

The goal of this announcement is to show that there is a principle that organizes many of these scaling exponents. We give numerical evidence that the exponents of different routes to chaos satisfy approximately some arithmetic relations. These relations are determined by combinatorial properties of the route and become exact in an appropriate limit.

PACS numbers: 05.45.Df 05.10.-a 05.10.Cc

1. Introduction

One of the most striking discoveries in dynamical systems in the last decades has been the existence of scaling relations and self-similarity in the transition to chaotic behavior. By now there is a plethora of transitions each with its own exponents. Many scaling relations have been explained by renormalization group (RG) theory [1]. Our goal here is to report on some organization among the scaling exponents of different transitions.

In many transitions, one can associate a combinatorics to the maps. For example, in unimodal maps we can prescribe the kneading sequence or in quasiperiodic maps the rotation number. There are natural operations among these combinatorics. For example, one can use the $*$-operation among finite kneading sequences [2] or just the juxtaposition among finite continued fraction expansions.

If we fix one of these combinatorics, we can consider successions of bifurcations obtained by selecting mappings whose combinatorics is a power of the operation. For example, the standard period doubling corresponds to considering $(R*)^n$ and the usual quasi-periodic route for the golden mean to take continued fractions $F_n/F_{n+1} = [1, \ldots, 1]$. If we choose other sequence to raise to increasing powers, we obtain different routes to chaos, which are often found to have scaling properties.

In this short paper we formulate the *Principle of Approximate Combination of Scaling Exponents* (PACSE for short). PACSE asserts that the scaling exponents of different transitions are related. PACSE can be formulated briefly as follows.

(A) If the combinatorics of two routes to chaos are joined, by the natural operation, the scaling exponents of the joint combinatorics are approximated by the product of the exponents of the two original routes.

(B) The approximate product rule in (A) becomes exact if the combination is repeated infinitely many times.

(C) The convergence in part (B) is exponential.

In Sections 2–4, we make explicit the meaning of the combinatorics and their combination rules in some examples. We also present numerical evidence for the points (A), (B), (C) above. In Section 5 we indicate some theoretical ideas why PACSE should be true.

2. PACSE for critical circle maps

2.1. Real- and parameter-space scaling exponents

We first describe the principle of approximate combination in the case of quasiperiodic route to chaos.

In [3, 4, 5, 6] there is a description of phenomena in families of maps with a cubic critical point and rotation number equal to the *golden mean*, $\sigma_{\mathrm{G}} := \frac{1}{2}(\sqrt{5} - 1)$.

In this paper we consider more general quasi-periodic transitions. The different quasi-periodic transitions are characterized by an irrational number $\rho \in [0, 1)$ called the *rotation number*. The combinatorics of the transition is given by the continued fraction expansion (CFE) of ρ [7]. We write $\langle a_1, a_2, \ldots\rangle = 1/(a_1 + 1/(a_2 + \cdots))$. We consider numbers whose CEF is eventually periodic. Given finite sequences of natural numbers $\mathsf{A} = (a_1, a_2, \ldots, a_p)$ and $\mathsf{B} = (b_1, b_2, \ldots, b_q)$, we define their *concatenation* $\mathsf{AB} = (a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_q)$ and denote repeated concatenations by by exponents.

We will denote by $\langle \mathsf{AB}^\infty\rangle = \langle \mathsf{ABB}\ldots\rangle$ the irrational number whose CFE is the concatenation of A and infinitely many copies of B. To such a number, we can associate a sequence of rational numbers which converge to it, $\frac{P_n}{Q_n} = \langle \mathsf{AB}^n\rangle$.

As prototypes of circle maps we consider the two-parameter family

$$f_{\omega,\beta}(x) = [x + \omega + \beta\, g(x)] \mod 1 \ , \tag{1}$$

where $g(x)$ is a smooth periodic function of period 1, i.e., $\sin 2\pi x$. If $f'(x)$ becomes 0 at one point c (and, therefore, f^{-1} is not differentiable), we say that f is a *critical map* and call c the *critical point* of f. Below $f^n(x)$ will stand for the nth *iteration*, i.e., to the map f applied n times: $f^n(x) = f \circ f \circ \cdots \circ f(x)$.

For many rotation numbers of the form $\langle \mathsf{AB}^\infty\rangle$ the following behavior has been observed.

(a) Parameter-space scaling. For a fixed value of the nonlinearity parameter β in (1), let $I_n(\beta)$ be a *phase locking interval*, i.e., the interval of values of the parameter ω for which the circle map $f_{\omega,\beta}$ has rational rotation number $\frac{P_n}{Q_n} = \langle \mathsf{AB}^n\rangle$. Then the lengths of the phase locking intervals behave with n as

$$|I_n(\beta)| \approx C\delta_{\mathsf{B}}^{-n} \ ,$$

where δ_{B} is a universal number, that is, a number that depends only on B (but not on the head A of CFE $\langle \mathsf{AB}^\infty\rangle$) and on the order of the critical point c, but is otherwise independent of the families $f_{\omega,\beta}$ (when the families range over a small enough neighborhood). We will indicate the order of the critical point c with a superscript on δ_{B}.

(b) Real-space scaling (scaling of recurrences). Let f be a critical circle map with critical point c, and rotation number $\langle \mathsf{AB}^\infty\rangle$. Then the iterates $f^{Q_n}(c)$ approach c geometrically:

$$|f^{Q_n}(c) - c| \approx C\alpha_{\mathsf{B}}^{-n} \ ,$$

where α_{B} is a universal number (in the same sense as for δ_{B}). A superscript will denote the order of criticality.

2.2. Formulation of PACSE for critical circle maps

For circle maps, PACSE is expressed as follows.

(i) For a fixed order of criticality of c, there exist constants C_1 and C_2 such that

$$C_1 \le \frac{\delta_{\mathsf{AB}}}{\delta_{\mathsf{A}}\,\delta_{\mathsf{B}}} \le C_2 \ , \qquad C_1 \le \frac{\alpha_{\mathsf{AB}}}{\alpha_{\mathsf{A}}\,\alpha_{\mathsf{B}}} \le C_2 \ , \tag{2}$$

where C_1 and C_2 depend only on $\max(a_1, \ldots, a_p, b_1, \ldots, b_q)$.

(ii) For a fixed order of criticality of c and fixed A and B, the following limits exist:

$$\lim_{k\to\infty} \frac{\delta_{\mathsf{A}^k\mathsf{B}}}{(\delta_{\mathsf{A}})^k\,\delta_{\mathsf{B}}}\ , \qquad \lim_{k\to\infty} \frac{\alpha_{\mathsf{A}^k\mathsf{B}}}{(\alpha_{\mathsf{A}})^k\,\alpha_{\mathsf{B}}}\ . \tag{3}$$

(iii) For a fixed order of criticality of c and fixed A and B, the ratios $\mathscr{D}_k = \frac{\delta_{\mathsf{A}^k\mathsf{B}}}{(\delta_{\mathsf{A}})^k\delta_{\mathsf{B}}}$ and $\mathscr{A}_k = \frac{\alpha_{\mathsf{A}^k\mathsf{B}}}{(\alpha_{\mathsf{A}})^k\alpha_{\mathsf{B}}}$ approach their limiting values $\mathscr{D}_\infty$ and $\mathscr{A}_\infty$ exponentially:

$$|\mathscr{D}_k - \mathscr{D}_\infty| \approx C\xi^k\ , \qquad |\mathscr{A}_k - \mathscr{A}_\infty| \approx C\eta^k\ , \tag{4}$$

for some constants ξ and η.

The bounds (2) are quite surprising because α_{A} and especially δ_{A} are huge when the length of A is large.

2.3. Evidence for PACSE for critical circle maps

We studied the following families of circle maps.

• The "standard" cubic critical (C) family ($0 \leq K < \frac{4}{3}$)

$$f^{\mathrm{C}}_{K,\omega}(x) = \left[x + \omega - \tfrac{1}{2\pi}(K \sin 2\pi x + \tfrac{1-K}{2}\sin 4\pi x)\right] \bmod 1\ ,$$

where the coefficients are chosen in such a way that for every K in the interval $(0, \frac{4}{3})$, $f^{\mathrm{C}}_{K,\omega}(x) = \omega + \frac{2\pi^2(4-3K)}{3}x^3 + \mathcal{O}(x^5)$.

• The "standard" quintic critical (Q) family ($\frac{1}{2} \leq K < \frac{3}{2}$)

$$f^{\mathrm{Q}}_{K,\omega}(x) = \left[x + \omega - \tfrac{1}{2\pi}(K \sin 2\pi x + \tfrac{9-8K}{10}\sin 4\pi x + \tfrac{3K-4}{15}\sin 6\pi x)\right] \bmod 1\ ,$$

where the coefficients are chosen in such a way that for every K in the interval $(\frac{1}{2}, \frac{3}{2})$, $f^{\mathrm{Q}}_{K,\omega}(x) = \omega + \frac{8\pi^4(3-2K)}{5}x^5 + \mathcal{O}(x^7)$.

• The maps

$$f(x) = x + \omega - \frac{b}{2\pi}\,\frac{\sin 2\pi x}{a - \cos 2\pi x} \tag{5}$$

with $(a, b) = (2, 1)$, for which f is cubic critical $[f(x) = \omega + \frac{8}{3}\pi^2 x^3 + \mathcal{O}(x^5)]$; and $(a, b) = (-2, -3)$, for which f is quintic critical $[f(x) = \omega + \frac{4}{45}\pi^4 x^5 + \mathcal{O}(x^7)]$.

Since the maps (5) with a fixed order of critical point do not contain any free parameter, we only computed the α exponents. This was to reassure us that the results apply to functions with infinitely harmonics.

For more detail on algorithms for computing the critical values, we refer to [8], which only considered the case of the golden mean. Table 1 presents our numerical results for the scaling exponents δ and α for the cases of cubic and quintic critical maps. In the quintic critical case we studied rotation numbers of the form $\langle (1^k 2)^\infty \rangle$, while in the cubic critical case we did it for $\langle (1^k 2)^\infty \rangle$ and $\langle (1^k 3)^\infty \rangle$. The errors do not exceed 2 in the last digit. When we did not detect good enough convergence we just put "?".

In Table 2 we present the ratios in (2) and (3), keeping only the number of digits

Table 1. Scaling exponents of C and Q maps with rotation numbers $\langle \mathsf{B}^\infty \rangle$, $\mathsf{B} = (1^k 2)$, and of C maps with rotation numbers $\langle \mathsf{B}^\infty \rangle$, $\mathsf{B} = (1^k 3)$.

	Cubic critical, $\mathsf{B} = (1^k 2)$		Cubic critical, $\mathsf{B} = (1^k 3)$		Quintic critical, $\mathsf{B} = (1^k 2)$	
k	$\delta^{\mathrm{C}}_{1^k 2}$	$\alpha^{\mathrm{C}}_{1^k 2}$	$\delta^{\mathrm{C}}_{1^k 3}$	$\alpha^{\mathrm{C}}_{1^k 3}$	$\delta^{\mathrm{Q}}_{1^k 2}$	$\alpha^{\mathrm{Q}}_{1^k 2}$
0	6.79922516	1.58682670	13.760284	1.855060	7.7912246	1.3791501
1	17.66905276	1.9691355	31.623877	2.17411	21.573320	1.5985
2	52.04449	2.590589	98.32467	2.945324	68.620816	1.9392
3	145.425152	3.308635	269.104	3.71001	205.43	2.2997
4	414.51561	4.28301	774.04	4.836423	629.5	2.7536
5	1171.7123	5.5067	2179.3	6.19630	1910.6	3.2836
6	3323.73	7.1039	6193	8.0082	5820	3.9216
7	9413.7	9.14860	17530	10.3035	17710	4.6815
8	26681	11.7923	$49700_{\pm 20}$	13.287	53500	5.590
9	75590	15.1929	140800	17.117	160000	6.677
10	214000	19.579	$400000_{\pm 20000}$	22.061	500000	7.970
11	607900	25.230	?	28.428	1600000	9.53

Table 2. Ratios of scaling exponents as in (2) for the data from Table 1.

	Cubic critical, $\mathsf{B} = (1^k 2)$		Cubic critical, $\mathsf{B} = (1^k 3)$		Quintic critical, $\mathsf{B} = (1^k 2)$	
k	$\dfrac{\delta^{\mathrm{C}}_{1^k 2}}{\left(\delta^{\mathrm{C}}_1\right)^k \delta^{\mathrm{C}}_2}$	$\dfrac{\alpha^{\mathrm{C}}_{1^k 2}}{\left(\alpha^{\mathrm{C}}_1\right)^k \alpha^{\mathrm{C}}_2}$	$\dfrac{\delta^{\mathrm{C}}_{1^k 3}}{\left(\delta^{\mathrm{C}}_1\right)^k \delta^{\mathrm{C}}_3}$	$\dfrac{\alpha^{\mathrm{C}}_{1^k 3}}{\left(\alpha^{\mathrm{C}}_1\right)^k \alpha^{\mathrm{C}}_3}$	$\dfrac{\delta^{\mathrm{Q}}_{1^k 2}}{\left(\delta^{\mathrm{Q}}_1\right)^k \delta^{\mathrm{Q}}_2}$	$\dfrac{\alpha^{\mathrm{Q}}_{1^k 2}}{\left(\alpha^{\mathrm{Q}}_1\right)^k \alpha^{\mathrm{Q}}_2}$
1	0.9170936095	0.96302277	0.81104983	0.909524	0.90981984	0.97084
2	0.9533118	0.9832182	0.8899277	0.9562160	0.95091785	0.98652
3	0.940068655	0.9745199	0.859552	0.934735	0.93539	0.97995
4	0.94562895	0.978997	0.87252	0.9456449	0.9418	0.98283
5	0.94332356	0.97682	0.86694	0.940214	0.93926	0.98170
6	0.944333	0.97793	0.8694	0.94302	0.9401	0.98206
7	0.94389	0.977369	0.8685	0.941586	0.9300	0.98199
8	0.94411	0.977671	0.869	0.94231	0.924	0.9822
9	0.9439	0.977519	0.869	0.94207	0.93	0.9827
10	0.943	0.97761	0.87	0.94226	0.94	0.9825
11	0.945	0.97765	?	0.94229	1.0	0.984

such that the error does not exceed 2 in the last digit, unless otherwise specified. In the computations we used the values of the scaling exponents δ_1 and α_1 for rotation number golden mean, $\langle 1^\infty \rangle$: $\delta^{\mathrm{C}}_1 = 2.8336106559$, $\alpha^{\mathrm{C}}_1 = 1.28857456$, $\delta^{\mathrm{Q}}_1 = 3.04337774$, $\alpha^{\mathrm{Q}}_1 = 1.193857$.

As a numerical confirmation of the exponential convergence of the ratios $\mathscr{D}_k$ and $\mathscr{A}_k$ (see (4)), we present in Figure 1 the differences $|\mathscr{D}_k - \mathscr{D}_\infty|$ and $|\mathscr{A}_k - \mathscr{A}_\infty|$ versus k. Note that for large values of k the computation is very sensitive to numerical errors.

We also studied numerically the scaling exponents for maps with rotation numbers that are eventually periodic (like $\langle 31^\infty \rangle$, $\langle 3^2 1^\infty \rangle$, $\langle 3^2 (12)^\infty \rangle$), and found that the values of the scaling exponents depend only on the tail of the continued fraction expansions,

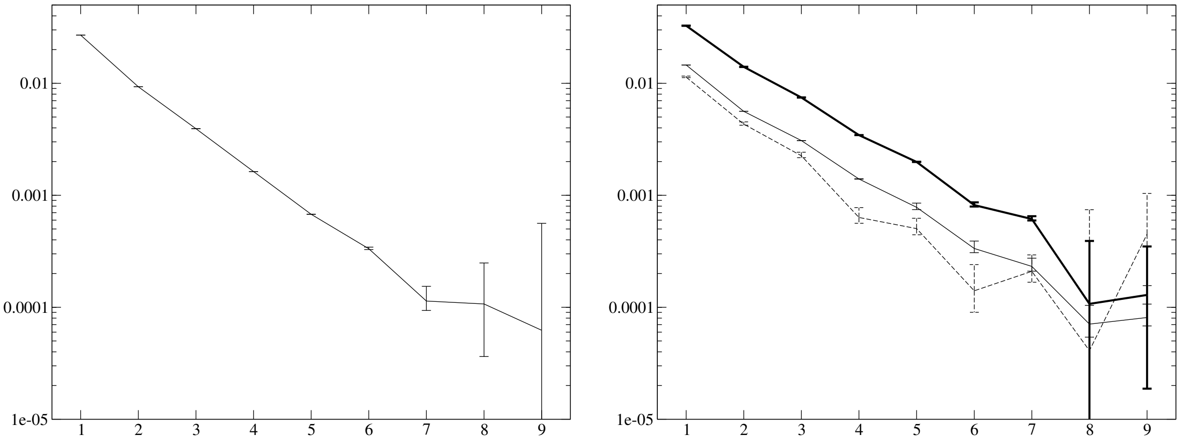

Figure 1. **Left:** log-linear plot of the differences $\left|\delta^{\mathrm{C}}_{1^k2}/[(\delta^{\mathrm{C}}_1)^k\delta^{\mathrm{C}}_2] - 0.944\right|$; **right:** log-linear plot of the differences $\left|[\alpha^{\mathrm{C}}_{1^k2}/[(\alpha^{\mathrm{C}}_1)^k\alpha^{\mathrm{C}}_2] - 0.9776\right|$ (thin lines), $\left|[\alpha^{\mathrm{C}}_{1^k3}/(\alpha^{\mathrm{C}}_1)^k\alpha^{\mathrm{C}}_3] - 0.9422\right|$ (thick lines), $|[\alpha^{\mathrm{Q}}_{1^k2}/(\alpha^{\mathrm{Q}}_1)^k\alpha^{\mathrm{Q}}_2] - 0.9822|$ (dashed lines) vs. k for $k = 1, 2, \ldots, 9$.

as predicted by the general theory.

3. PACSE for area-preserving twist maps of the cylinder

We considered the standard (Taylor-Chirikov) family of area-preserving twist maps,

$$(x', y') = (x + y', y + \tfrac{K}{2\pi}\sin 2\pi x) \ .$$

We refer to [9] for background on the standard maps and the algorithms used, although [9] studies only one fixed rotation number. RG theory of such maps is developed in [10, 6, 11, 12].

Given a rotation number $\langle \mathsf{B}^\infty\rangle$, the critical events we studied are the existence of tangencies between the stable and unstable manifolds of periodic orbits with rotation number $\langle \mathsf{B}^n\rangle$ and $\langle \mathsf{B}^{n+1}\rangle$. Denoting the critical parameter values by K_n, and the area of the lobes enclosed by the tangency by L_n, we observed the following scalings:

$$|K_n - K_\infty| \approx C\Delta_{\mathsf{B}}^{-n} \ , \qquad L_n \approx C\lambda_{\mathsf{B}}^{-n} \ ,$$

where Δ_{B} and λ_{B} are universal numbers depending only on B. In Table 3 we summarize our numerical results.

4. Other contexts for PACSE

In this section, we describe some other contexts for PACSE, making precise the meaning of combinatorics. We have some numerical data, but will not present it to keep the announcement short.

Table 3. Scaling exponents of critical area-preserving twist maps with rotation numbers $\langle (21^k)^\infty \rangle$; we used that $\Delta_1 = 1.62802$, $\Delta_2 = 2.4569$, $\lambda_1 = 4.339143$, $\lambda_2 = 14.60$.

k	Δ_{21^k}	λ_{21^k}	$\frac{\Delta_{21^k}}{\Delta_2\,(\Delta_1)^k}$	$\frac{\lambda_{21^k}}{\lambda_2\,(\lambda_1)^k}$
1	3.778	59.548	0.944	0.939
2	6.298	284.53	0.967	1.035
3	10.168	1141.44	0.959	0.956

4.1. Unimodal maps of the interval

Renormalization for unimodal maps is analogous to the renormalization for circle maps. The kneading sequences play a role similar to the role of the rotation numbers. An analogue of concatenation of continued fractions is the $*$-operation of [13, 14, 2, 15]. For a given a kneading sequence K and a family f_λ, it is standard to check for the parameters λ_n characterized by the critical point being periodic and having an itinerary given by K^{*n}. In this case, one can obtain parameter- and real-space scaling exponents by

$$|\lambda_n - \lambda_\infty| \approx C\delta_{\mathsf{K}}^{-n}, \qquad f_{\lambda_{n+1}}^{|\mathsf{K}^{*n}|}(0) \approx C\alpha_{\mathsf{K}}^{-n} .$$

Again, δ_{K}, α_{K} are universal numbers which depend on K. If $\mathsf{K} = \mathsf{K}_1 * \mathsf{K}_2$, PACSE predicts that

$$\delta_{\mathsf{K}} \approx \delta_{\mathsf{K}_1}\delta_{\mathsf{K}_2} , \qquad \alpha_{\mathsf{K}} \approx \alpha_{\mathsf{K}_1}\alpha_{\mathsf{K}_2} .$$

4.2. Boundaries of Siegel disks

We recall that Siegel disks are the domains of stability around the origin of maps of the complex plane of the form $f(z) = az + O(z^2)$ where $a = \exp(2\pi i \langle \mathsf{B}^\infty \rangle)$.

Siegel disks have been intensively studied from the renormalization point of view since [16, 17]. It has been found that, in many cases, the boundaries of the Siegel disks contain a critical point c. The renormalization theories are very similar to that of rotations since multiplication by a is just a rotation, so that the combinatorics are the same as those of the circle maps. We can define real-space scaling exponents by $f^{Q_n}(c) - c \approx C\alpha_{\mathsf{B}}^{-n}$ and parameter-space scaling by searching for a_n such that $f^{Q_n}(c) = c$ and verifying $a_n - a \approx C\delta_{\mathsf{B}}^{-n}$, where again α_{B} and δ_{B} are universal but depend on B. In contrast with the other cases mentioned above, the scaling exponents are complex numbers. Nevertheless, for each level of renormalization, the scaling exponents conjugate.

We have that

$$|\delta_{\mathsf{AB}}| \approx |\delta_{\mathsf{A}}|\,|\delta_{\mathsf{B}}| , \qquad |\alpha_{\mathsf{AB}}| \approx |\alpha_{\mathsf{A}}|\,|\alpha_{\mathsf{B}}| .$$

4.3. Rigid rotations and smooth diffeomorphisms of the circle

For rigid rotations of the circle PACSE follows from a detailed study [18] of the Gauss map [7]. For circle diffeomorphisms PACSE for real-space scalings follows from the fact that the map can be smoothly conjugated to a rigid rotation [19, 20].

4.4. p-renormalization

In [21], the authors considered some special kneading sequences $\mathsf{K} = RLLL \dots L$, for which they could compute rigorously the asymptotics of the scaling exponents. By extending slightly their computation, one can verify PACSE both for parameter- and real-space scaling [18].

4.5. Unimodal maps that are functions of $|x|^{1+\epsilon}$

The papers [22, 14] consider unimodal maps that are functions of $|x|^{1+\epsilon}$ and construct fixed points of renormalization. They present calculations of some fixed points and their scaling exponents for ϵ small. By extending their calculations slightly, one can verify [18] the first statement of PACSE up to leading order in ϵ. Indeed, some version of PACSE for parameter-space scaling is mentioned in passing in [14, p. 276].

5. Conclusions

We established numerically some approximate relations between several scaling exponents of different RGs.

The existence of this regularity seems to be good evidence that there is a global RG applicable for maps of all rotations (or all kneading sequences). It seems possible that PACSE can be considered evidence for certain dynamical behaviors of these global renormalization operators. The paper [18] suggests that PACSE is evidence for the existence of a horseshoe with 1-dimensional unstable manifolds which, furthermore, satisfy some transversality conditions.

Global renormalizations have been proposed for Siegel discs [23, 24, 25, 26], unimodal maps [27, 28], critical circle maps [29, 30]. We hope that this paper can serve as a stimulus for the development of these theories.

Acknowledgments

The work of RdlL and NPP was partially supported by NSF grants.

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