Q1]. . . [16 points] Evaluate the following trigonometric integrals. Show all your work.

$$\int \sin^2(x)\,dx$$

$$\int \sin^2(x)\,dx \;=\; \frac{1}{2}\int (1-\cos(2x))\,dx \;=\; \frac{1}{2}(x - \frac{\sin(2x)}{2}) \,+\, C$$

$$\int \sec^3(x)\tan^2(x)\,dx$$

Let $u = \sec(x) + \tan(x)$ and $v = \sec(x) - \tan(x)$. Recall that $uv = 1$, and $\sec(x) = (u+v)/2$, and $\tan(x) = (u-v)/2$. Recall also that $\sec(x)dx = du/u = -dv/v$. We get

$$\int \;=\; \int ((u+v)/2)^2((u-v)/2)^2 \sec(x)dx \;=\; \frac{1}{16}\int (u^2-v^2)^2 \sec(x)dx$$

Since $uv = 1$ the square term becomes $u^4 - 2 + v^4$, and so the integral becomes

$$\frac{1}{16}\int (u^4-2)\frac{du}{u} \;-\; \frac{1}{16}\int v^4\frac{dv}{v} \;=\; \frac{1}{64}u^4 - \frac{1}{8}\ln|u| - \frac{1}{64}v^4 + C$$

Q2]. . . [16 points] Evaluate the following integrals.

$$\int x^2 e^x\,dx$$

Use integration by parts with $u = x^2$ and $dv = e^x dx$ so that $du = 2xdx$ and $v = e^x$.

$$\int x^2 e^x dx \;=\; x^2e^x - 2\int xe^x dx$$

We evaluate the last integral above by parts again with $u = x$ and $dv = e^x dx$ so $du = dx$ and $v = e^x$. This gives

$$\int x^2 e^x dx \;=\; x^2e^x - 2(xe^x - \int e^x dx) = x^2e^x - 2xe^x + 2e^x + C$$

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx$$

Let $x = 3\sin(\theta)$ so that $dx = 3cos(\theta)d\theta$ and $\sqrt{9-x^2} = 3\cos(\theta)$. We get

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx \;=\; \int \frac{(3\sin(\theta))^2 3\cos(\theta)d\theta}{3\cos(\theta)} \;=\; 9\int \sin^2(\theta)d\theta$$

We did this integral in Question 1 above. Thus the original integral is equal to

$$\frac{9\theta}{2} - \frac{9\sin(2\theta)}{4} + C \;=\; \frac{9\theta}{2} - \frac{9(2\sin(\theta)\cos(\theta))}{4} + C \;=\; \frac{9}{2}\sin^{-1}(\frac{x}{3}) - \frac{1}{2}x\sqrt{9-x^2} + C$$

Q3]. . . [9 points] Find constants A, B, C which make the following a true statement.

$$\frac{1-x}{x^3+9x} \;=\; \frac{A}{x} \;+\; \frac{Bx+C}{x^2+9}$$

Writing the RHS as a single fraction, and comparing numerators gives

$$A(x^2+9) + Bx^2 + Cx \;=\; 1 - x$$

Thus

$$A + B = 0, \qquad C = -1, \qquad 9A = 1$$

and so $A = 1/9$, $B = -1/9$ and $C = -1$.
Using your answer above, evaluate the integral

$$\int \frac{1-x}{x^3+9x}\,dx$$

By the partial fractions algebra above, we can rewrite the integral as

$$\frac{1}{9}\int \frac{dx}{x} - \frac{1}{9}\int \frac{xdx}{x^2+9} - \int \frac{dx}{x^2+9}$$

The first term is a log integral, the second term is a log integral (after making the substitution $u = x^2 + 9$ and $du/2 = xdx$), and the third term is an arctan integral. Thus we get

$$\int \frac{1-x}{x^3+9x}\,dx \;=\; \frac{1}{9}\ln|x| \;-\; \frac{1}{18}\ln(x^2+9) \;-\; \frac{1}{3}\tan^{-1}(\frac{x}{3}) \;+\; C$$

Q4]... [9 points] Show that

$$\cosh^2(x) \;-\; \sinh^2(x) \;=\; 1$$

By definition of $\cosh(x)$ and $\sinh(x)$, the left hand side is equal to

$$((e^x+e^{-x})/2)^2 \;-\; ((e^x-e^{-x})/2)^2 \;=\; \frac{e^{2x}+2+e^{-2x}-(e^{2x}-2+e^{-2x})}{4} \;=\; \frac{2+2}{4} \;=\; 1$$

Show that

$$\frac{d}{dx}\sinh(x) \;=\; \cosh(x)$$

By definition of $\cosh(x)$ and $\sinh(x)$ we get that the left side is equal to

$$\frac{d}{dx}\frac{e^x-e^{-x}}{2} \;=\; \frac{e^x-e^{-x}(-1)}{2} \;=\; \frac{e^x+e^{-x}}{2}$$

which is equal to the right side.
Now determine the derivative

$$\frac{d}{dx}\sinh^{-1}(x)$$

The function $y = \sinh^{-1}(x)$ can be rewritten as $x = \sinh(y)$. We differentiate this implicitly with respect to x to get

$$1 = \frac{dx}{dx} \;=\; \frac{d\sinh(y)}{dx} \;=\; \frac{d\sinh(y)}{dy}\frac{dy}{dx} \;=\; \cosh(y)\frac{dy}{dx}$$

Solving for $\frac{dy}{dx}$ gives us

$$\frac{d}{dx}\sinh^{-1}(x) \;=\; \frac{dy}{dx} \;=\; \frac{1}{\cosh(y)} \;=\; \frac{1}{\sqrt{\cosh^2(y)}} \;=\; \frac{1}{\sqrt{1+\sinh^2(y)}} \;=\; \frac{1}{\sqrt{1+x^2}}$$

1. Trig Addition.

   $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$
   $\cos(2A) = \cos^2(A) - \sin^2(A)$
   $\cos(2A) = 2\cos^2(A) - 1$
   $\cos(2A) = 1 - 2\sin^2(A)$
   $\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$.

2. Hyperbolic.

   $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$
   $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$

3. Integration by Parts.

   $\int u\,dv \;=\; uv \;-\; \int v\,du$

4. Inverse Trig.

   $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$
   $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$
   $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a})$

5. Jon McCammond Dictionary.

   $u = \sec(x) + \tan(x)$ and $v = \sec(x) - \tan(x)$
   $\sec(x) = \frac{u+v}{2}$, and $\tan(x) = \frac{u-v}{2}$, and $uv = 1$
   $\sec(x)dx = \frac{du}{u}$ and $\sec(x)dx = \frac{-dv}{v}$

6. Trig Substitutions.

   For $\sqrt{a^2 - x^2}$ use $x = a\sin(\theta)$
   For $\sqrt{a^2 + x^2}$ use $x = a\tan(\theta)$
   For $\sqrt{x^2 - a^2}$ use $x = a\sec(\theta)$