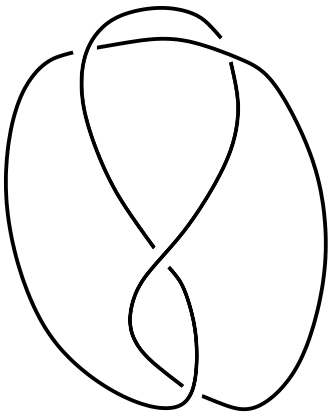
Math 5863-001 Final Examination Name:______________________

Friday, May 9, 2003, 8:00am–10:00am.

Answer as many questions as you can (stomach).

Q1].. Write down a Wirtinger presentation for $\pi_1(S^3 \setminus \text{Fig-8 knot})$.



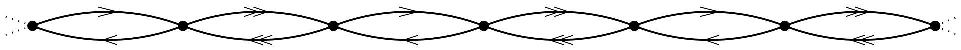
Determine (showing your work) the abelianization of $\pi_1(S^3 \setminus \text{Fig-8 knot})$.

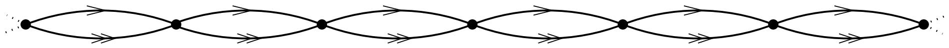
Prove that there does not exist a retraction from the space X (which is defined to be S^3 minus an open tubular neighborhood of the Fig-8 knot) to the boundary torus of X.

Q2].. Use covering spaces (as in the proof of the Kurosh theorem) to give a complete description of the kernel of the homomorphism

$$\psi : \langle a \,|\, a^2 \rangle * \langle b \,|\, b^3 \rangle \;\rightarrow\; \mathrm{Perm}(\{1,2,3\}) \;\; : a \mapsto (12), \;\; : b \mapsto (123)$$

Q3].. Here are two infinite regular covering spaces of the bouquet of two circles. Determine the automorphism group in each case. [Say how the automorphisms act on the covering spaces, and say why you have listed all of the automorphisms in each case]





Q4].. State the Path Homotopy Lifting Property (PHLP) and the Path Lifting Property (PLP) for covering spaces $p : \tilde{X} \to X$.

Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a path-connected covering space. Let $G = \pi_1(X, x_0)$ and $H = p_*(\pi_1(\tilde{X}, \tilde{x_0}))$. Prove (sketch) that the map

$$\Phi : p^{-1}(x_0) \to G/H \ : \tilde{x_1} \mapsto H[p \circ \gamma]$$

where γ is a path in $\tilde{X}$ from $\tilde{x_0}$ to $\tilde{x_1}$, is well-defined, injective, and surjective. Conclude that the cardinality of $p^{-1}(x_0)$ equals the index of H in G.

Well-defined

Injective [This requires a little thought]

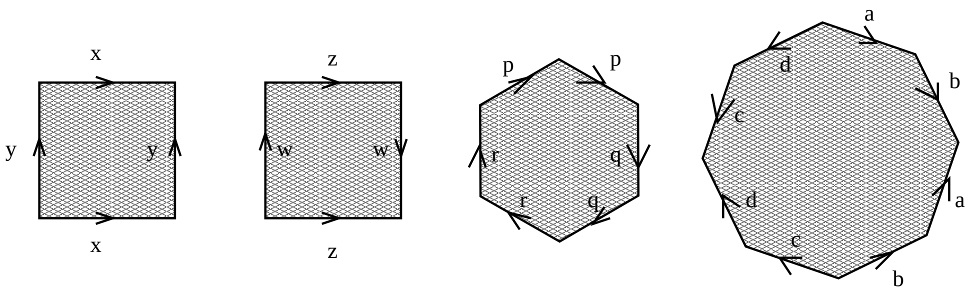
Surjective

Q5].. By constructing a covering space of the bouquet of two circles, give a detailed description of the subgroup H of the free group $F_{\{a,b\}}$ which is generated as follows:

$$\langle\, aba,\, baa,\, abba\bar{b},\, b\bar{a}\bar{b}a \,\rangle$$

- Determine the index of H in $F_{\{a,b\}}$.
- Is H free (if so describe a free basis for H)?
- Is H normal in $F_{\{a,b\}}$?
- Check whether $a^3b^3 \in H$.

Q6].. Use van Kampen to write down a presentation of the fundamental groups of the following four surfaces.



Use algebra to distinguish between the 4 groups. Are any two of these surfaces homeomorphic to each other?

Q7].. Use van Kampen's theorem to determine the effect on π_1 when you remove an open ball from an n-manifold ($n \geq 3$).

Give an example to show what can happen when $n = 2$.

Q8].. The first two parts help you to answer the remaining parts.

- Write down the relationship between $\pi_1(X \times Y, (x_0, y_0))$ and $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$.

- Write down an explicit homeomorphism between $\mathbb{R}^n \setminus \{(0, \ldots, 0)\}$ and $S^{n-1} \times (0, \infty)$.

- What is $\pi_1(\mathbb{R}^3 \setminus x\text{-axis})$?

- What is $\pi_1(\mathbb{R}^3 \setminus (x\text{-axis} \cup y\text{-axis}))$?

- What is $\pi_1(\mathbb{R}^4 \setminus x\text{-axis})$?

- What is $\pi_1(\mathbb{R}^4 \setminus xy\text{-plane})$?

- What is $\pi_1(\mathbb{R}^4 \setminus (xy\text{-plane} \cup zw\text{-plane}))$?