Q1 $\hat{r} = \langle \cos\theta, \sin\theta \rangle \Rightarrow \frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} = \langle -\sin\theta, \cos\theta \rangle \dot{\theta}$

$$= \dot{\theta}\,\hat{\theta}$$

$\hat{\theta} = \langle -\sin\theta, \cos\theta \rangle \Rightarrow \frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt} = -\langle \cos\theta, \sin\theta \rangle \dot{\theta}$

$$= -\dot{\theta}\,\hat{r}$$

Q2 $\dot{\vec{r}} = \frac{d(r\hat{r})}{dt} = \dot{r}\,\hat{r} + r\frac{d\hat{r}}{dt}$

$$= \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}$$ .. by Q1.

Q3 $\ddot{\vec{r}} = \frac{d}{dt}\left(\dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}\right)$

sum & Product rules

$$= \frac{d(\dot{r})}{dt}\hat{r} + \dot{r}\left(\frac{d\hat{r}}{dt}\right) + \frac{dr}{dt}\dot{\theta}\,\hat{\theta} + r\frac{d\dot{\theta}}{dt}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$= \left(\ddot{r} - r(\dot{\theta})^2\right)\hat{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta}$$

.. by Q1

Q4 We're told Kepler's 1st Law that motion is in a plane.

So we can work with $\hat{r}, \hat{\theta}$ coordinates in this plane (assuming it's the xy-plane, in xyz space).

Then (from Q2) ...

$$\vec{r} \times \dot{\vec{r}} = (r\hat{r}) \times (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$$

$$= r\dot{r}(\hat{r} \times \hat{r}) + r^2\dot{\theta}(\hat{r} \times \hat{\theta})$$

$$= \vec{0} + r^2\dot{\theta}\hat{k} \qquad \boxed{\hat{r} \times \hat{\theta} = \hat{k}}$$

Proven in class
(or you can reprove here)

By Kepler II, we know $r^2\dot{\theta} = 2C$ .... (constant)

so $\boxed{\vec{r} \times \dot{\vec{r}} = 2C\hat{k}}$

Q5

$$\frac{d}{dt}(\vec{r} \times \dot{\vec{r}}) = \cancel{\frac{d}{dt}(2C\hat{k})} = \vec{0}$$

constant.

$$\Rightarrow \frac{d\vec{r}}{dt} \times \dot{\vec{r}} + \vec{r} \times \frac{d\dot{\vec{r}}}{dt} = \vec{0}$$

$$\underbrace{\cancel{\dot{\vec{r}} \times \dot{\vec{r}}}}_{0} + \vec{r} \times \ddot{\vec{r}} = \vec{0}$$

$$\vec{r} \times \ddot{\vec{r}} = \vec{0} \Rightarrow \ddot{\vec{r}} \parallel \vec{r}$$

$$\Rightarrow m\ddot{\vec{r}} \parallel \vec{r} \Rightarrow \boxed{\text{central Force}}$$

$\boxed{\vec{F} = m\ddot{\vec{r}}}$ Newton Law of motion

Q6 (This was essentially written out in the problem set) . . .

So we can write $\vec{F}(r) = m\vec{a}$

$$= m\ddot{\vec{r}}$$

$$= m\left(\ddot{r} - r(\dot{\theta})^2\right)\hat{r} + m\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta}$$

$f(r)\hat{r}$

Central force (by Q5)

$\Rightarrow$ No $\hat{\theta}$ component.

by Q3

We're told $\hat{\theta}$ component $= 0$

$$\Rightarrow \boxed{\vec{F} = f(r)\hat{r} = m\left(\ddot{r} - r(\dot{\theta})^2\right)\hat{r}}$$

& we can compute $f(r)$ from $m\left(\ddot{r} - r\dot{\theta}^2\right)$ . . . .

Q7 Kepler I $\Rightarrow$ $r = \dfrac{\varepsilon d}{1+\varepsilon\cos\theta}$

$$\Rightarrow \dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \left(\frac{-\varepsilon d}{(1+\varepsilon\cos\theta)^2}(-\varepsilon\sin\theta)\right)\dot{\theta}$$

$$\dot{r} = \frac{\varepsilon^2 d\sin\theta}{(1+\varepsilon\cos\theta)^2}\dot{\theta} = \frac{2C\varepsilon^2 d\sin\theta}{(1+\varepsilon\cos\theta)^2 r^2}$$

by Kepler II

$$\left(\dot{\theta} = \frac{2C}{r^2}\right)$$

But $\quad r^2 = \dfrac{(\varepsilon d)^2}{(1+\varepsilon\cos\theta)^2}$

$$\Rightarrow \quad \dot r = \frac{2C\,\cancel{\varepsilon^2}\,\cancel{d}\sin\theta}{\cancel{(1+\varepsilon\cos\theta)^2}} \cdot \frac{\cancel{\varepsilon^2 d^2}}{\cancel{(1+\varepsilon\cos\theta)^2}}$$

$$= \frac{2C\sin\theta}{d}$$

$$\Rightarrow \quad \ddot r = \frac{d\dot r}{d\theta}\frac{d\theta}{dt} = \frac{d}{d\theta}\left(\frac{2C}{d}\sin\theta\right)\dot\theta$$

$$= \frac{2C}{d}\cos\theta \,.\, \dot\theta = \frac{2C}{d}\cos\theta\,\frac{2C}{r^2}$$

Finally $\quad f(r) = m\left(\ddot r - r(\dot\theta)^2\right) \quad \ldots$ from <u>Qb</u>.

$$= m\left(\frac{2C}{d}\cos\theta\,\frac{2C}{r^2} - r\left(\frac{2C}{r^2}\right)^2\right)$$

$$= \frac{4C^2 m}{r^2}\left(\frac{\cos\theta}{d} - \frac{1}{r}\right)$$