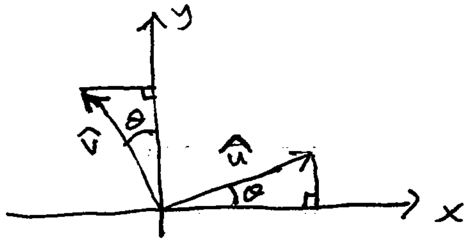
MATH 2433 – Extra II

Q1



$$\hat{u} = \langle \cos\theta, \sin\theta \rangle$$

$$\hat{v} = \langle -\sin\theta, \cos\theta \rangle$$

Use trig functions, keep eye on signs.

Q2

$$\begin{aligned}\vec{P} &= u\hat{u} + v\hat{v} \\ &= u\langle \cos\theta, \sin\theta\rangle + v\langle -\sin\theta, \cos\theta\rangle \\ &= \langle u\cos\theta - v\sin\theta,\ u\sin\theta + v\cos\theta\rangle \\ &= x\hat{\imath} + y\hat{\jmath}\end{aligned}$$

$$\left.\begin{aligned} x &= u\cos\theta - v\sin\theta \\ y &= u\sin\theta + v\cos\theta \end{aligned}\right] \text{——} (*)$$

Q3 $\boxed{\text{Case } C=0}$ $Ax^2 + By^2 + Dx + Ey + F = 0$

Assuming $A \neq 0$, $B \neq 0$ $\Rightarrow$

$$A\left(x^2 + 2\frac{D}{2A}x\right) + B\left(y^2 + 2\frac{E}{2B}y\right) + F = 0$$

$$A\left(x + \frac{D}{2A}\right)^2 + B\left(y + \frac{E}{2B}\right)^2 + F = \frac{D^2}{4A} + \frac{E^2}{4B}$$

$$A\left(x + \frac{D}{2A}\right) + B\left(y + \frac{E}{2B}\right)^2 = \underbrace{\frac{D^2}{4A} + \frac{E^2}{4B} - F}_{\text{call this } G}$$

CASE $A = B$ $\Rightarrow$ CIRCLE (want G & A, B to have same sign).

CASE A, B same sign $\Rightarrow$ ellipse (want G, A, B all to have same sign).

A, B opposite signs $\Rightarrow$ HYPERBOLA (No cond^{n} on G).

If $B=0$ $\quad A\left(x+\frac{D}{2A}\right)^2 + Ey = \frac{D^2}{4A} - F$ parabola.

Likewise, if $A=0$ $(B\neq 0)$, get parabola.

<u>Q4.</u> [CASE $C\neq 0$] By $(*)$ we get (subst for x, y)

$$A(u\cos\theta - v\sin\theta)^2 + B(u\sin\theta + v\cos\theta)^2 + C(u\cos\theta - v\sin\theta)(u\sin\theta + v\cos\theta)$$
$$+ D(u\cos\theta - v\sin\theta) + E(u\sin\theta + v\cos\theta) + F = 0$$

This becomes

$$\left[A\cos^2\theta + B\sin^2\theta + C\cos\theta\sin\theta\right]u^2 + \left[A\sin^2\theta + B\cos^2\theta - C\cos\theta\sin\theta\right]v^2$$

$$+\left[-2A\cos\theta\sin\theta + 2B\cos\theta\sin\theta + C(\cos^2\theta - \sin^2\theta)\right]uv + \left[D\cos\theta + E\sin\theta\right]u +$$

$$\left[E\cos\theta - D\sin\theta\right]v + F = 0$$

Coefficient of the uv term is --- using $\left[\begin{array}{l} 2\cos x\sin x = \sin(2x) \\ \cos^2 x - \sin^2 x = \cos(2x) \end{array}\right]$ ---

$$-(A-B)\sin(2\theta) + C\cos(2\theta)$$

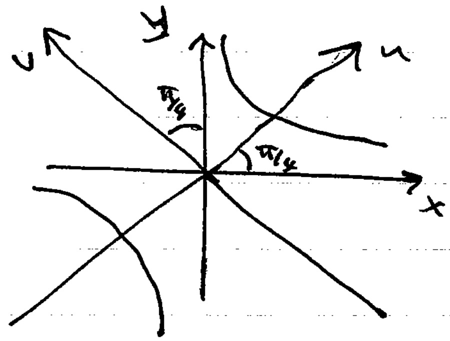
This coefficient is $0 \Leftrightarrow$ $\boxed{\tan(2\theta) = \frac{C}{A-B}}$ — $(**)$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{C}{A-B}\right)$$

~~So~~ After rotating so uv-axes make angle $\theta = \frac{1}{2}\tan^{-1}\left(\frac{C}{A-B}\right)$ with xy-axes the new equation has no uv term.

$\Rightarrow$ [Conic by Q3].

Q5 (example) $\quad y = \frac{1}{x}$



$$xy = 1$$

$C = 1, \quad A = B = 0 \qquad \theta = \frac{1}{2}\tan^{-1}(\text{undef.})$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

$(*) \Rightarrow \quad x = \frac{1}{\sqrt{2}}u - \frac{1}{\sqrt{2}}v \qquad y = \frac{1}{\sqrt{2}}u + \frac{1}{\sqrt{2}}v$

$xy = 1$ becomes $\quad \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(u-v)(u+v) = 1$

$$\frac{u^2}{2} - \frac{v^2}{2} = 1 \qquad \text{hyperbola!}$$

PART 2 — (Ants).

At time t ants are at $\vec{A}(t)$, $\vec{B}(t)$, $\vec{C}(t)$; $\vec{D}(t)$

[†] — [Ants will be coplanar $\iff$ $(\vec{B}(t)-\vec{A}(t)) \cdot [(\vec{C}(t)-\vec{A}(t)) \times (\vec{D}(t)-\vec{A}(t))]$ is ZERO.]

We have to show RHS of (†) is zero at some time

Define $f(t) = (\vec{B}(t)-\vec{A}(t)) \cdot [(\vec{C}(t)-\vec{A}(t)) \times (\vec{D}(t)-\vec{A}(t))]$

$$= \begin{vmatrix} (b_1(t)-a_1(t)) & (b_2(t)-a_2(t)) & (b_3(t)-a_3(t)) \\ (c_1(t)-a_1(t)) & (c_2(t)-a_2(t)) & (c_3(t)-a_3(t)) \\ (d_1(t)-a_1(t)) & (d_2(t)-a_2(t)) & (d_3(t)-a_3(t)) \end{vmatrix}$$

Note since $a_i(t), b_i(t), c_i(t), d_i(t)$ are 12 continuous functions, and $f(t)$ = sum of products of differences of those 12 functions, then $f(t)$ is a continuous function of t.

If we can show f changes sign — [† †]

then Intermediate Value Theorem $\Rightarrow$ $\exists t$ such that $f(t) = 0$

$\Rightarrow$ done!

We'll prove [++] by showing $f(0)$ & $f(1)$ have opposite signs.

$$f(0) = \left(\vec{B}(0) - \vec{A}(0)\right) \cdot \left[\left(\vec{C}(0) - \vec{A}(0)\right) \times \left(\vec{D}(0) - \vec{A}(0)\right)\right]$$

$$= \left(\vec{P}_2 - \vec{P}_1\right) \cdot \left[\left(\vec{P}_3 - \vec{P}_1\right) \times \left(\vec{P}_4 - \vec{P}_1\right)\right]$$

$$= \left(\vec{P}_2 - \vec{P}_1\right) \cdot \left[\vec{P}_3 \times \vec{P}_4 - \vec{P}_3 \times \vec{P}_1 - \vec{P}_1 \times \vec{P}_4 + \cancelto{\vec{0}}{\vec{P}_1 \times \vec{P}_1}\right]$$

$$= \vec{P}_2 \cdot \left(\vec{P}_3 \times \vec{P}_4\right) - \vec{P}_2 \left(\vec{P}_3 \times \vec{P}_1\right) - \vec{P}_2 \cdot \left(\vec{P}_1 \times \vec{P}_4\right)$$

$$- P_1 \cdot \left(P_3 \times P_4\right) + 0 + 0$$

---- $(P_3 \times P_1) \perp P_1$

$(P_1 \times P_4) \perp P_1$

$$f(1) = \left(\vec{B}(1) - \vec{A}(1)\right) \cdot \left[\left(\vec{C}(1) - \vec{A}(1)\right) \times \left(\vec{D}(1) - \vec{A}(1)\right)\right]$$

= same expression as for $f(0)$ with $P_1 \to P_2 \to P_3 \to P_4 \to P_1$.

$$= \vec{P}_3 \cdot \left(\vec{P}_4 \times \vec{P}_1\right) - \vec{P}_3 \cdot \left(\vec{P}_4 \times \vec{P}_2\right) - \vec{P}_3 \cdot \left(\vec{P}_2 \times \vec{P}_1\right)$$

$$- \vec{P}_2 \cdot \left(\vec{P}_4 \times \vec{P}_1\right)$$

$$= \vec{P}_1 \cdot \left(\vec{P}_3 \times \vec{P}_4\right) - \vec{P}_2 \cdot \left(\vec{P}_3 \times \vec{P}_4\right) - \vec{P}_2 \cdot \left(\vec{P}_1 \times \vec{P}_3\right)$$

$$+ \vec{P}_2 \cdot \left(\vec{P}_1 \times \vec{P}_4\right)$$

---- use $A \cdot (B \times C) = C \cdot (A \times B)$ for 1st 3 terms & $(A \times B) = -(B \times A)$ on 4th term.

Now we can easily see that $f(1) = -f(0) \Rightarrow$ done!!